Classification of Reductive Groups (Kaletha-Prasad §10.7)

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Splitting Lemma: $N \trianglelefteq G \curvearrowright X$ s.t. $N \curvearrowright X$ is a torsor.
Then $\forall x \in X$, $G_x \xrightarrow{\sim} G/N$ and $G = N \rtimes G_x$:

$$\begin{tikzcd}
1 \arrow[r] & N \arrow[r] & G \arrow[r] & G/N \arrow[r] \arrow[l, bend right, dotted] & 1 \\
 & & & G_x \arrow[u, "\simeq"'] \arrow[ul] &
\end{tikzcd}$$

Proof: Easy exercise.

Applications:

N	G	X
$W(\Psi)$	$W(\Psi)^{ext}$	chambers of Ψ
$\mathrm{Inn}(G)$	$\mathrm{Aut}(G)$	pinnings
$W(\Phi)$	$\mathrm{Aut}(\Phi)$	bases of Φ
Ξ	$\mathrm{Aut}(\mathrm{Dyn}(\Psi))$	special points of $\mathrm{Dyn}(\Psi)$

Digression on Dynkin diagrams

Draw root systems beforehand!

Start w/ Lemma...

Setup: Ψ affine root system, $\Phi = \nabla \Psi$ root system

Assume Ψ is irreducible and reduced $\Rightarrow \Psi = \Psi_\Phi$ or $\Psi^\vee_{\Phi^\vee}$.

$\Delta \subset \Phi$ basis, $\tilde{\alpha} \in \Phi$ longest root. Affine bases:

- $\Delta \cup \{1 - \tilde{\alpha}\}$ for Ψ_Φ
- $\Delta \cup \{1 - \tilde{\alpha^\vee}^\vee\}$ for $\Psi^\vee_{\Phi^\vee}$. $\quad \left(\mathrm{Dyn}(\Psi^\vee) = \mathrm{Dyn}(\Psi)^\vee \right)$

Conversely, given $\tilde{\Delta} \subset \Psi$ basis, $\exists\ \delta \in \tilde{\Delta}$ s.t.

$$\mathrm{Dyn}(\Psi, \tilde{\Delta}) \setminus \{\delta\} \simeq \mathrm{Dyn}(\Phi).$$

We call such δ <u>special</u>.

Automorphisms

$$1 \longrightarrow W(\Phi) \longrightarrow \mathrm{Aut}(\Phi) \longrightarrow \mathrm{Out}(\Phi) \longrightarrow 1$$

$$\mathrm{Out}(\Phi) \xleftarrow{\ \simeq\ } \mathrm{Aut}(\Phi, \Delta) \simeq \mathrm{Aut}(\mathrm{Dyn}(\Phi)), \qquad \mathrm{Aut}(\Phi, \Delta) \longrightarrow \mathrm{Aut}(\Phi)$$

Same diagram for Ψ, C.

Claim: $\exists$ finite ab. gp. Ξ s.t.

$$\mathrm{Aut}(\mathrm{Dyn}(\Psi)) \simeq \Xi \rtimes \mathrm{Aut}(\mathrm{Dyn}(\Phi)).$$

Proof:

$$\begin{tikzcd}
 & & Q^\vee \arrow[d, equal] & P^\vee \arrow[d, equal] & & \\
 & 1 \arrow[r] & \mathbb{Z}\Psi^\vee \arrow[r] \arrow[d] & (\mathbb{Z}\Psi)^* \arrow[r] \arrow[d] & P^\vee/Q^\vee \arrow[r] \arrow[d] & 1 \\
 & 1 \arrow[r] & W(\Psi) \arrow[r] \arrow[d] & \mathrm{Aut}(\Psi) \arrow[r] \arrow[d] & \mathrm{Aut}(\mathrm{Dyn}(\Psi)) \arrow[r] \arrow[d, dashed] & 1 \\
\nabla \downarrow & 1 \arrow[r] & W(\Phi) \arrow[r] & \mathrm{Aut}(\Phi) \arrow[r] & \mathrm{Aut}(\mathrm{Dyn}(\Phi)) \arrow[r] & 1
\end{tikzcd}$$

Fact (Bourbaki): $P^\vee/Q^\vee \curvearrowright \{x \in \mathrm{Dyn}(\Psi) \text{ special}\}$ simply transitively.

$$\begin{tikzcd}
1 \arrow[r] & P^\vee/Q^\vee \arrow[r] & \mathrm{Aut}(\mathrm{Dyn}(\Psi)) \arrow[r] & \mathrm{Aut}(\mathrm{Dyn}(\Phi)) \arrow[r] \arrow[l, bend right, dashed] & 1 \\
 & & & \mathrm{Aut}(\mathrm{Dyn}(\Psi), x). \arrow[u, "\simeq"'] \arrow[ul] &
\end{tikzcd}$$

$\Xi = P^\vee/Q^\vee$. $\square$

Alternatively:

Def: $W(\Psi)^{ext} := \{ f \in \mathrm{Aut}(\Psi) : \nabla f \in W(\Phi)\}$, extended Weyl gp.

$W(\Psi)^{ext} \curvearrowright \mathrm{Aut}(\mathrm{Dyn}(\Psi))$ w/ kernel $W(\Psi)$

$$\Rightarrow P^\vee/Q^\vee \simeq W(\Psi)^{ext}/W(\Psi) \simeq W(\Psi)^{ext}_c.$$

Example:

	Φ	Ξ	$\mathrm{Aut}(\mathrm{Dyn}(\Phi))$
	A_n	$\mathbb{Z}/(n+1)$	$\mathbb{Z}/2$
$(n\neq 2)$	D_{2n}	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$ $(n\neq 2)$ or S_3 $(n=2)$
	D_{2n+1}	$\mathbb{Z}/4\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
	E_6	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$

Folding (Ψ arbitrary now)

$$\Gamma \subset \mathrm{Aut}(\Psi, C) \rightsquigarrow \Psi_\Gamma \quad \text{"folded" root system}$$

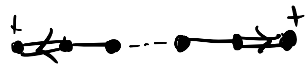
$$\mathrm{Aut}(\Psi, C) \cong \mathrm{Aut}(\mathrm{Dyn}(\Psi))$$

$$\mathrm{Dyn}(\Psi_\Gamma) = \mathrm{Dyn}(\Psi)/\Gamma$$

(w/ rules for weights, arrows)

Example: $\mathbb{Z}/2\mathbb{Z} \curvearrowright D_{n+1} \rightsquigarrow B_n$

$\mathbb{Z}/3\mathbb{Z} \curvearrowright \tilde{E}_6 \rightsquigarrow \tilde{G}_2^\vee$

$\mathbb{Z}/2\mathbb{Z} \curvearrowright \tilde{A}_{2n-1} \rightsquigarrow (C_n^\vee, C_n)$ 

Every nonsimply laced affine root system can be obtained by folding.

Classification of reductive gps

$\bar{k}$ = sep. cl. of k.

Forms

$$\text{split reductive gps}/k \xrightarrow{\simeq} \text{root data}$$

$(\exists T \subseteq G$ split max. torus$)$

$$\mathcal{R} = (\Phi \subset X^*(T), \Phi^\vee \subset X_*(T))$$

root system + integral structure

Fix G/k (split).

$$\psi \longmapsto (\sigma \mapsto \psi^{-1} \circ \sigma\psi)$$

$$\{G'/k + \psi : G_{\bar{k}} \simeq G'_{\bar{k}}\}/\sim \xrightarrow{\simeq} H^1(k, \mathrm{Aut}(G))$$

$$G_z \longleftarrow z$$

Inner forms

(Fr: épinglage (Grothendieck))

A <u>pinning</u> of G is $(T \subseteq B, \{0 \neq X_\alpha \in U_\alpha\}_{\alpha \in \Delta})$.

$\mathcal{B}$ (or Φ) is a basis.

G is <u>quasisplit</u> if it has a pinning (equiv., a Borel).

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{Inn}(G) \arrow[r] \arrow[d, equal] & \mathrm{Aut}(G) \arrow[r] & \mathrm{Out}(G) \arrow[r] \arrow[l, bend right, dotted] & 1 \\
& G/Z =: G_{ad} & & \mathrm{Aut}(G, T \subseteq B, \{X_\alpha\}) \arrow[ul] \arrow[u, "\simeq"'] \arrow[d, "\simeq"] & \\
& & & \mathrm{Aut}(\mathcal{R}_G, \Delta) &
\end{tikzcd}$$

Example: $G = T$ torus $\Rightarrow$ $\mathrm{Out}(G) = GL(X^*(T))$

$G = G_{ad}$ $\Rightarrow$ $\mathrm{Out}(G) = \mathrm{Aut}(\mathrm{Dyn}(G))$

(may be smaller even if $G = G_{der}$)

Corollary: ① Fix G^s/k split.

$$\{G/k \text{ quasisplit s.t. } G_{\bar{k}} \simeq G^s_{\bar{k}}\} \simeq H^1(k, \mathrm{Out}(G^s))$$
$$\simeq H^1(k, \mathrm{Aut}(R_G, \Delta))$$

② More generally,

$$\{G/k \text{ quasisplit}\}/\sim \;\simeq \{\text{pinned root data} + \text{pinned } \mathrm{Gal}(\bar{k}/k)\text{-action}\}.$$

Corollary: Split SES of pointed sets

$$1 \longrightarrow H^1(k, G_{ad}) \longrightarrow H^1(k, \mathrm{Aut}(G)) \longrightarrow H^1(k, \mathrm{Out}(G)) \longrightarrow 1$$

Def: G' is an <u>inner form</u> of G if $G' \simeq G_z$ for $z \in Z^1(k, G_{ad})$.

$$\{\text{inner forms of } G\}/\sim \;\simeq H^1(k, G_{ad}).$$

$$\{G/k \text{ reductive}\}/\sim \;\simeq \{(G^*/k \text{ quasisplit}, G/k \text{ inner form of } G^*\}/\sim$$

Example: $H^1(k, PGL_n) \simeq \{A/k \text{ central simple algebra}, [A:k] = n^2\}$

$$\updownarrow$$

$\underline{A^\times}/k$

Forms in Bruhat–Tits theory

k as in Kaletha–Prasad, $\dim(\mathfrak{f}) \leq 1$, $\Gamma = \mathrm{Gal}(K/k)$,

Heuristic: Iwahoris act like Borels, so G/k acts like it is quasisplit.

Last time: $C \subset \mathcal{B}(G)$ chamber; T/k special s.t. $C \subset \mathcal{A}(T)$.

$$\begin{aligned} H^1(k, G_{ad}) &\simeq H^1(K/k, N_{G_{ad}(K)}(C) / G_{ad}(K)_C^{\circ}) \\ &\simeq H^1(K/k, N_{N_{G_{ad}}(T_{ad})(K)}(C) / T_{ad}(K)) \\ &\simeq H^1(K/k, \Xi_C) \end{aligned}$$

Recall: $\Xi_C = W(\Psi)^{ext}_C \simeq \mathrm{Aut}(\mathrm{Dyn}(\Psi)) / \mathrm{Aut}(\mathrm{Dyn}(\Phi))$.

$G \rightsquigarrow f : \Gamma \longrightarrow \mathrm{Aut}(\Phi_K)$ (Tits's $*$-action)

$$H^1(K/k, \Xi) \simeq \left\{ \begin{tikzcd} \tilde{f} : \Gamma \arrow[r] \arrow[dr, "f"'] & \mathrm{Aut}(\mathrm{Dyn}(\Psi_K)) \arrow[d] \\ & \mathrm{Aut}(\mathrm{Dyn}(\Psi_K)) \end{tikzcd} \text{ s.t. } \right\}$$

Theorem: $\{G/k \text{ reductive}\}/\sim \;\longleftrightarrow\; \{(\text{root data } \mathcal{R} \curvearrowleft \mathrm{Gal}(\bar{k}/k), \Gamma \curvearrowright \Psi_K \text{ lifting } \Gamma \curvearrowright \Phi_K)\}$.

<u>Description of root systems</u> $G \longleftrightarrow (G^*/k \text{ quasisplit}, \tilde{f})$.

$$G^* \rightsquigarrow \Phi_K = \Phi(G^*, S^*) \cap \Gamma, \quad S^* \subset G^* \text{ max. split}/K$$

$\wr$ fold $\qquad\qquad \cap|$

$$\Phi = \Phi(G^*, T^*) \cap \mathrm{Gal}(\bar{k}/k) \quad T^* \qquad \text{minimal Levi}$$

<u>Thm</u>: <u>Assume</u> Φ_K reduced + irreducible. Then

$$\Psi_K = \begin{cases} \Psi_{\Phi_K} & \text{if } G^*_K \text{ is split} \\ \Psi^\vee_{\Phi^\vee_K} & \text{if not.} \end{cases}$$

As for G, have $\quad \Phi_k = \Phi(G, S), \quad S \subseteq G$ max. split

$$\Psi_k = \Psi(G, S)$$

<u>Thm</u>: $\Psi_K \rightsquigarrow \Psi_k$ by folding.

<u>Example</u>: ① Say $G^* \simeq GL_n$, k local ($\Rightarrow \Gamma \simeq \hat{\mathbb{Z}}$).

Inner forms of GL_n are $GL_m(D)$, $\qquad D/k$ div. alg., $\dim_k D = m^2$, $m \cdot n = D$.

$\Gamma \curvearrowright \mathrm{Aut}^\circ(\tilde{A}_{n-1}) \simeq \mathbb{Z}/n\mathbb{Z}$ (draw wheel).

Recover class field theory fact: $\mathrm{Br}(F)[n] \simeq \mathbb{Z}/n\mathbb{Z}$.

② Groups/k of type E_8, F_4, and G_2 are split.

<u>Corollary</u>: k local, $G = G_{ad}$. $G(k)$ compact $\Rightarrow G \simeq \underline{D}^\times/\underline{k}^\times$.