Hubbard trees for post-singularly finite exponential maps

Michael Rothgang

BMS Student Conference, February 22, 2017
Bachelor's research project,
at Jacobs University Bremen
Thesis advisor: Prof. Dierk Schleicher

Outline of my talk

1. What do we investigate?
2. How does a combinatorial classification work?
3. Known classification results
4. How to (not) construct Hubbard trees

What do we study?

- holomorphic dynamics, links complex analysis + dyn. systems
  iteration of entire holomorphic functions

What do we study?

- holomorphic dynamics, links complex analysis + dyn. systems iteration of entire holomorphic functions
- entire function $f : \mathbb{C} \to \mathbb{C} :=$ complex differentiable on $\mathbb{C}$
- *much* stronger than real differentiability!

What do we study?

- holomorphic dynamics, links complex analysis + dyn. systems
  iteration of entire holomorphic functions
- entire function $f : \mathbb{C} \to \mathbb{C}$:= complex differentiable on $\mathbb{C}$
- *much* stronger than real differentiability!
- for example, entire functions are analytic
- conversely, convergent power series define entire functions

$\Rightarrow$ entire function *is* a power series converging on $\mathbb{C}$

What is holomorphic dynamics? (cont.)

- iteration: given $z \in \mathbb{C}$, consider its orbit $z, f(z), f(f(z)), \ldots$

What is holomorphic dynamics? (cont.)

- iteration: given $z \in \mathbb{C}$, consider its orbit $z, f(z), f(f(z)), \ldots$
- goal: understand possible behaviour when iterating a function

What is holomorphic dynamics? (cont.)

- iteration: given $z \in \mathbb{C}$, consider its orbit $z, f(z), f(f(z)), \ldots$
- goal: understand possible behaviour when iterating a function
  Can points' orbits converge to infinity? Can they become periodic?

What is holomorphic dynamics? (cont.)

- iteration: given $z \in \mathbb{C}$, consider its orbit $z, f(z), f(f(z)), \ldots$
- goal: understand possible behaviour when iterating a function
  Can points' orbits converge to infinity? Can they become periodic? How common are these; how does a generic point behave?

What is holomorphic dynamics? (cont.)

- iteration: given $z \in \mathbb{C}$, consider its orbit $z, f(z), f(f(z)), \ldots$
- goal: understand possible behaviour when iterating a function
  Can points' orbits converge to infinity? Can they become periodic? How common are these; how does a generic point behave?
- Can we describe such behaviour in simpler terms?
- Classify functions by their possible behaviour

What is holomorphic dynamics? (cont.)

- iteration: given $z \in \mathbb{C}$, consider its orbit $z, f(z), f(f(z)), \ldots$
- goal: understand possible behaviour when iterating a function Can points' orbits converge to infinity? Can they become periodic? How common are these; how does a generic point behave?
- Can we describe such behaviour in simpler terms?
- Classify functions by their possible behaviour

How to classify the dynamics of entire functions

- different functions can have the same behaviour

$\Rightarrow$ want to classify behaviour, not the function itself!

How to classify the dynamics of entire functions

- different functions can have the same behaviour

$\Rightarrow$ want to classify behaviour, not the function itself!

- given a map, derive a combinatorial object from its dynamics (say a finite graph)
- characterise the combinatorial objects so obtained

How to classify the dynamics of entire functions

- different functions can have the same behaviour

$\Rightarrow$ want to classify behaviour, not the function itself!

- given a map, derive a combinatorial object from its dynamics (say a finite graph)
- characterise the combinatorial objects so obtained
- Does every such graph correspond to a map?
  Given a graph, construct a function having that graph!

How to classify the dynamics of entire functions

- different functions can have the same behaviour

$\Rightarrow$ want to classify behaviour, not the function itself!

- given a map, derive a combinatorial object from its dynamics (say a finite graph)
- characterise the combinatorial objects so obtained
- Does every such graph correspond to a map?
  Given a graph, construct a function having that graph!

Our big hammer: Thurston's theorem

Theorem (William Thurston, 1980s)

Every Thurston map is Thurston equivalent to a rational map iff there is no Thurston obstruction.

Our big hammer: Thurston's theorem

Theorem (William Thurston, 1980s)

Every Thurston map is Thurston equivalent to a rational map iff there is no Thurston obstruction.

- idea: combinatorial object $\rightarrow$ map on $\mathbb{S}^2 \leftrightarrow$ rational map

Our big hammer: Thurston's theorem

Theorem (William Thurston, 1980s)

Every Thurston map is Thurston equivalent to a rational map iff there is no Thurston obstruction.

- idea: combinatorial object $\rightarrow$ map on $\mathbb{S}^2$ $\leftrightarrow$ rational map

Our big hammer: Thurston's theorem

Theorem (William Thurston, 1980s)

Every Thurston map is Thurston equivalent to a rational map iff there is no Thurston obstruction.

- idea: combinatorial object $\to$ map on $\mathbb{S}^2$ $\leftrightarrow$ rational map

Our big hammer: Thurston's theorem

Theorem (William Thurston, 1980s)

Every Thurston map is Thurston equivalent to a rational map iff there is no Thurston obstruction.

- idea: combinatorial object $\rightarrow$ map on $\mathbb{S}^2$ $\leftrightarrow$ rational map

$\Rightarrow$ can find corresponding rational map
unless there is a very precise obstruction

Our big hammer: Thurston's theorem

Theorem (William Thurston, 1980s)

Every Thurston map is Thurston equivalent to a rational map iff there is no Thurston obstruction.

- idea: combinatorial object $\to$ map on $\mathbb{S}^2$ $\leftrightarrow$ rational map

$\Rightarrow$ can find corresponding rational map
unless there is a very precise obstruction

- reduces work to "just" a combinatorial-topological problem
- definitely avoiding a Thurston obstruction is still hard!

Which functions to classify?

- full classification is out of scope
  $\Rightarrow$ restrict to important and tractable classes of functions

Which functions to classify?

- full classification is out of scope
  $\Rightarrow$ restrict to important and tractable classes of functions

$\rightarrow$ (suitable classes of) polynomials

$\rightarrow$ rational functions = quotients of polynomials

$\rightarrow$ transcendental = non-polynomial entire functions

Post-singular set of an entire function f

- Can see f as a branched covering map $\mathbb{C} \to \mathbb{C}$
- Ramification points are called *singular values* of f

Post-singular set of an entire function f

- Can see f as a branched covering map $\mathbb{C} \to \mathbb{C}$
- Ramification points are called *singular values* of f
- *Post-singular set* $=$ (closure of) orbits of all singular values
- f is *post-singularly finite* iff its post-singular set is finite
  i.e. finitely many singular values, each is eventually periodic

Post-singular set of an entire function f

- Can see f as a branched covering map $\mathbb{C} \to \mathbb{C}$
- Ramification points are called *singular values* of f
- *Post-singular set* $=$ (closure of) orbits of all singular values
- f is *post-singularly finite* iff its post-singular set is finite
  i.e. finitely many singular values, each is eventually periodic
- for f polynomial, singular values $=$ critical values $=$ roots of f'
  also use the term post-critically finite

Post-singular set of an entire function f

- Can see f as a branched covering map $\mathbb{C} \to \mathbb{C}$
- Ramification points are called *singular values* of f
- *Post-singular set* = (closure of) orbits of all singular values
- f is *post-singularly finite* iff its post-singular set is finite
  i.e. finitely many singular values, each is eventually periodic
- for f polynomial, singular values = critical values = roots of f'
  also use the term post-critically finite
- why? describes most of the dynamics!

Known classification results

- post-critically finite polynomials: several classifications
- pcf rational functions: some further classes known, such as pcf Newton maps
- result for *all* pcf rational functions is current research!

Known classification results

- post-critically finite polynomials: several classifications
- pcf rational functions: some further classes known, such as pcf Newton maps
- result for *all* pcf rational functions is current research!
- post-singularly finite transcendental functions?

Known classification results

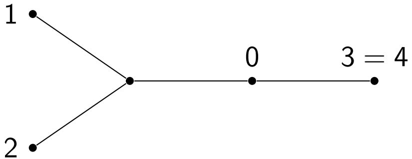
- post-critically finite polynomials: several classifications
- pcf rational functions: some further classes known, such as pcf Newton maps
- result for *all* pcf rational functions is current research!
- post-singularly finite transcendental functions? need Thurston's theorem!
- can classify psf exponential maps, but no more
- need good combinatorial invariants to move further

Our goal: build a new invariant

- construct combinatorial invariant for psf transcendental maps
- focus on exponential maps for simplicity
- Thurston's theorem proven for them $\rightarrow$ can classify
- hope: generalise to all psf transcendental maps

Our new invariant: Hubbard trees

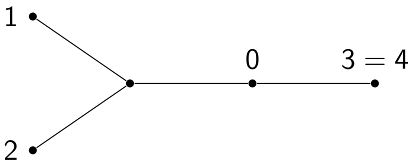
- Hubbard tree is a finite embedded tree
- vertices contain the post-singular set
- forward invariant: image is subset of tree again



Symbolic drawing of a Hubbard tree with post-singular set.

Our new invariant: Hubbard trees

- Hubbard tree is a finite embedded tree
- vertices contain the post-singular set
- forward invariant: image is subset of tree again



Symbolic drawing of a Hubbard tree with post-singular set.

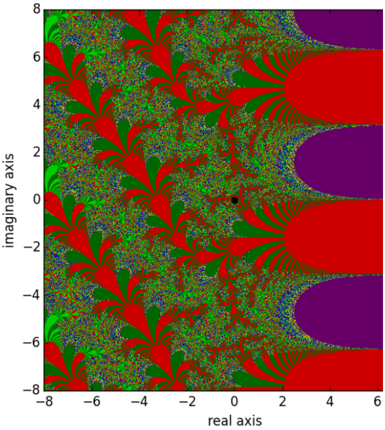
- for pcf polynomials, Hubbard trees exist
  essentially unique $\rightarrow$ can use for classification
- construct Hubbard trees for psf exponential maps

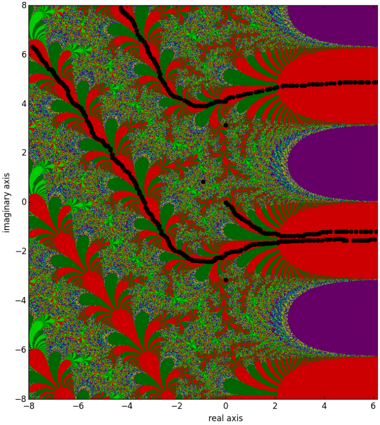
There cannot be Hubbard trees for psf exponential maps!

There cannot be Hubbard trees for psf exponential maps!

- consider *escaping set* $I(f) = \{z \in \mathbb{C} \colon f^n(z) \to \infty\}$
- theorem: is union of countably many disjoint continuous curves going to ∞
- *dynamic ray* = one such curve; does not self-intersect
- ray $\gamma : (0, \infty) \to \mathbb{C}$ lands at $a \in \mathbb{C}$ if $\gamma(t) \to a$ as $t \to 0$

There cannot be Hubbard trees for psf exponential maps!

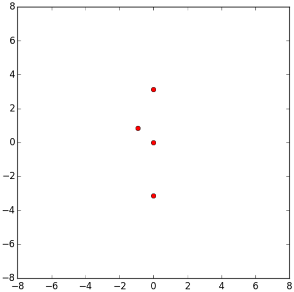




Some dynamic rays for the exponential map $i\pi \exp z$

There cannot be Hubbard trees for psf exponential maps!

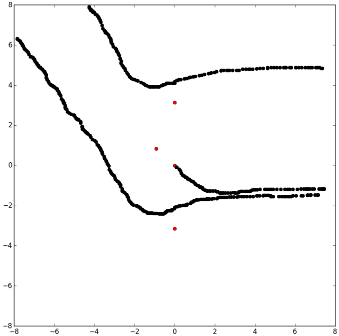
Example for $i\pi \exp z$



- recall definition: forward-invariant tree spanning post-singular set P

There cannot be Hubbard trees for psf exponential maps!

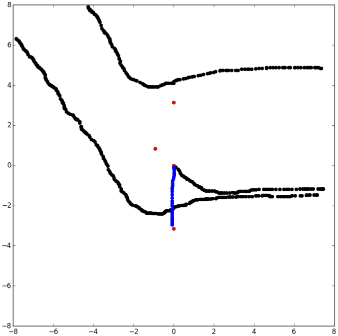
Example for $i\pi \exp z$



- recall definition: forward-invariant tree spanning post-singular set P
- exists dynamic ray g landing at singular value 0
- preimages of g disconnect $\mathbb{C}$ into countably many parts

There cannot be Hubbard trees for psf exponential maps!

Example for $i\pi \exp z$

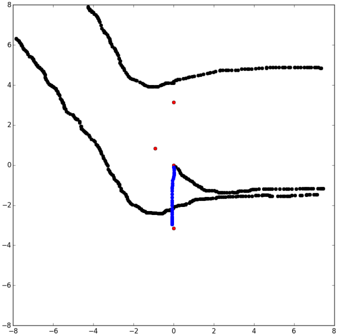


- recall definition: forward-invariant tree spanning post-singular set P
- exists dynamic ray g landing at singular value 0
- preimages of g disconnect $\mathbb{C}$ into countably many parts
- but: P spans several parts

$\Rightarrow$ Hubbard tree must cross a ray preimage, contradiction!

There cannot be Hubbard trees for psf exponential maps!

Example for $i\pi \exp z$



- recall definition: forward-invariant tree spanning post-singular set P
- exists dynamic ray g landing at singular value 0
- preimages of g disconnect $\mathbb{C}$ into countably many parts
- but: P spans several parts

$\Rightarrow$ Hubbard tree must cross a ray preimage, contradiction!

Solution: only require forward invariance *up to homotopy* rel vertices

Outline of tree construction

- there is a dynamic ray landing at 0
  using holomorphic dynamics, topology, hyperbolic geometry
- its preimages partition $\mathbb{C}$ into countably many parts/“sectors”
  $\rightsquigarrow$ each dynamic ray lies in exactly one sector
- find how further tree vertices must look like
  by symbolic dynamics and some graph theory
- these vertices exist: are landing points of dynamic rays
  (symbolic dynamics also)

$\Rightarrow$ Know the vertices our Hubbard tree must have

Outline of tree construction (cont.)

- tree edges?

Outline of tree construction (cont.)

- tree edges? "thou shalt not cross dynamic rays"
- every tree vertex has at least two dynamic rays landing
  choose tree edges as to avoid them

Outline of tree construction (cont.)

- tree edges? "thou shalt not cross dynamic rays"
- every tree vertex has at least two dynamic rays landing
  choose tree edges as to avoid them
- This determines how edges must run,
  there is an embedded tree that does not cross any such ray!
- last step: this tree candidate is indeed forward invariant

  all these: bit of topology, and classical discrete math

Next steps

- uniqueness of Hubbard trees
  Can we make our heuristic more rigorous?
- classification using Hubbard trees:
  given a tree, can we reconstruct the exponential map?
- extend to all post-singularly finite transcendental maps!

Thanks for your attention!

Better definitions of Hubbard trees: how to solve our problem

- define it away, consider e.g. cosine maps instead
- take trees crossing $-\infty$: will not work!
  every edge must contain $-\infty$, gives a contradiction
- relax forward invariance, allow deforming our edges (homotopy)

Why transcendental functions are much harder than polynomials

- point at ∞ behaves differently
- for polynomials, can set $f(\infty) = \infty$ in a nice way ("superattracting fixed point")
- transcendental maps, ∞ is an essential singularity!

Why transcendental functions are much harder than polynomials

- point at ∞ behaves differently
- for polynomials, can set $f(\infty) = \infty$ in a nice way ("superattracting fixed point")
- transcendental maps, ∞ is an essential singularity!
- defined function on $\mathbb{C} = \mathbb{S}^2 \setminus \{\infty\}$, what happens there?
  1. can extend f continuously $\Rightarrow$ must be constant
  2. f converges to infinity whenever $|z| \to \infty$, called a *pole* $\Rightarrow f$ is a polynomial or a rational function
  3. essential singularity: both finite and infinite limit values happen in every neighbourhood of ∞, f assumes all values in $\mathbb{C}$ (with at most two exceptions)

Why transcendental functions are much harder than polynomials

- point at ∞ behaves differently
- for polynomials, can set $f(\infty) = \infty$ in a nice way ("superattracting fixed point")
- transcendental maps, ∞ is an essential singularity!
- defined function on $\mathbb{C} = \mathbb{S}^2 \setminus \{\infty\}$, what happens there?
  1. can extend f continuously $\Rightarrow$ must be constant
  2. f converges to infinity whenever $|z| \to \infty$, called a *pole* $\Rightarrow f$ is a polynomial or a rational function
  3. essential singularity: both finite and infinite limit values happen in every neighbourhood of ∞, f assumes all values in $\mathbb{C}$ (with at most two exceptions)

Thus, transcendental dynamics is HARD. For example, escaping set for polynomials is homeomorphic to complement of a disc - for "many" transcendental maps, consists of countably many curves