Sapienza University of Rome

MATHEMATICAL, PHYSICAL AND NATURAL SCIENCES
Master's degree in Mathematics

A CLASS OF INFINITE ENERGY SOLUTIONS TO THE SUPERCRITICAL NLS

Advisor:
Prof.
Piero D'Ancona

Candidate:
Lorenzo Pompili
1714767

Summer session
Academic Year 2019-2020
Department of Mathematics 'Guido Castelnuovo'

A Class of Infinite Energy Solutions to the Supercritical NLS

Lorenzo Pompili

Mathematics Department G. Castelnuovo

Sapienza Rome University

2020

Summer session

Introduction

For many nonlinear models, the energy supercritical regime is still full of unresolved questions. Global existence for the defocusing NLS equation

$$i\partial_t v + \Delta v = +|v|^{\gamma-1}v$$

with large powers γ in the nonlinear term can be proven for arbitrary initial data with finite energy in low space dimensions 1 and 2, since in these cases the critical regularity index

$$s_c = \frac{d}{2} - \frac{2}{\gamma - 1}$$

is strictly less than one. On the other hand, in higher dimensions, global results can be proved if one assumes both high regularity and smallness of the initial datum. But even for data with high regularity and localization, it is still not clear if one should expect global well-posedness with large data (some very recent seps in this direction were made in [13] for systems with a suitable nonlinearity, and later in [10] for the classical defocusing equation).

A simple way to have distributional global solutions with 'large' data in $\mathbb{R}^d$ is to consider a 2D global solution u and treat it as a function $\widetilde{u}$ in $\mathbb{R}^d$ depending only on the first two variables. In this case, the initial datum $\widetilde{u}_0$ can be taken very regular supposing $u_0 \in H^m(\mathbb{R}^2)$, but it will clearly have infinite energy. Also, we don't have to suppose u_0 to have a small norm in $H^m(\mathbb{R}^2)$, i.e. there is an entire vector space of possible initial data whose solutions are global. In this thesis we study if it is possible to construct *small global* $\mathbb{R}^d$ *perturbations* of the essentially 2D solution $\widetilde{u}$, which depend on all variables, with high power γ, i.e. to achieve global well-posedness in a suitable sense for small perturbations φ of the initial data $\widetilde{u}_0$. As mentioned above, in the case $\varphi = 0$ this fact is well established, and it is reasonable to conjecture that the same holds for perturbations of $\tilde{u}$, provided it satisfies suitable decay conditions for large times.

This work is organized as follows. In Chapter 1, we prove preliminary results for the two-dimensional solution, recalling and discussing some well known facts about the defocusing NLS equation. The main new result in this part of the thesis is a decay estimate of the L^∞ norm of the 2D solution $\tilde{u}$, as a preparation to perturbed high dimensional equation around the second part; this is proved through a suitable application of the pseudo-conformal transform.

In Chapter 2, we establish the local theory in H^m, m, for the perturbed equation around the 2D solution $\tilde{u}$. We assume here $\frac{d}{2} < m \in \mathbb{N}$ as a regularity assumption, which was comfortable while proving the main theorem in the early stages, but not necessary for the purpose of it. Recently, I tried to deal with the more general regularity assumption $s_c < m \in \mathbb{N}$, i.e. the general subcritical case, and it seems to work well. Nonetheless, due to lack of time, the local theory in this case has not been developed in detail, and is just discussed in Chapter 4 as a possible development of this work.

In Chapter 3, we find a global *a priori* bound for the solution in the hypothesis $\varphi \in H^m(\mathbb{R}^d)$, $m > \frac{d}{2}$, and with suitable regularity assumptions on u_0. Thence, a global solution exists thanks to the blowup criterion. The proof relies on standard tools from Schrödinger theory (*Strichartz estimates* and nonlinear estimates). Besides, we show that the proof works also in the subcritical case $m > s_c$, provided local well-posedness still holds.

In Chapter 4, we first discuss local well-posedness in the subcritical case; we also list some ideas about possible improvements of this result with similar techniques. We then conduct a preliminary study on the natural generalization of this problem, i.e. when u_0 depends on n variables, $n < d$. We show that this new problem can be reasonably faced with the same techniques only in the cases $n = 1, 2$ and $d \leq 4$, plus some sporadic cases.

Contents

Chapter 1

Unperturbed problem

The problem we are formally going to study in this work is the defocusing pure-power nonlinear Schrödinger equation with an unusual intial datum:

$$\begin{cases} iv_t + \Delta v = |v|^{\gamma-1}v & \text{in } I \times \mathbb{R}^d, \\ v(0,\mathbf{x}) = \widetilde{u}_0(\mathbf{x}) + \varphi(\mathbf{x}), \end{cases} \tag{1.1}$$

where $I \subseteq \mathbb{R}$ is an open interval, $\mathbf{x} \in \mathbb{R}^d$, $d \geq 3$, $\gamma > 1$ odd or big enough to make the nonlinearity regular, $\widetilde{u}_0$ depends only on the first two variables and φ is a small perturbation that depends on all of them. Calling $x \in \mathbb{R}^2$ the first two components of $\mathbf{x}$ and $z \in \mathbb{R}^{d-2}$ the remaining ones, so that $\mathbf{x} = (x, z)$, we can write

$$\widetilde{u}_0(x, z) \coloneqq u_0(x).$$

for a 2D function u_0. Our goal throughout the work is to develop a local theory for the problem and then prove global existence for small initial data $\varphi \in H^m(\mathbb{R}^d)$ and general $u_0 \in H^s(\mathbb{R}^2)$, for suitable $m \in \mathbb{N}_0$ and $s \in \mathbb{R}_{\geq 0}$.

We will see a solution to this Cauchy problem as a perturbation of the case $\varphi = 0$. In order to do that, we will consider the classical nonlinear Schrödinger model

$$\begin{cases} i\partial_t u + \Delta_x u = |u|^{\gamma-1}u, & (t,x) \in I \times \mathbb{R}^n, \\ u(0,x) = u_0(x). \end{cases} \tag{NLS}$$

for $n = 2$. If u is a solution to (NLS), we can call $\widetilde{u}(x,z) \coloneqq u(x)$ and consider v as a perturbation of the extended solution $\widetilde{u}$. In fact, it's clear that $\widetilde{u}$ is at least a distributional solution of problem (1.1) with $\varphi = 0$.

Remark 1.0.1. It's not immediate to say in what sense $\widetilde{u}$ is a solution of (1.1) with $\varphi = 0$ and its uniqueness in some space, but we will not be interested in this, since we just want to study perturbations of this solution. However, this is of course the most natural way to define 'the' solution, once we have a unique solution u of (NLS) in a suitable space, as we will see in a few pages.

The main theorem we want to prove is Theorem 3.0.1. The result is written in terms of the perturbation $w := v - \widetilde{u}$, which will be introduced in Chapter 2, so we will leave the details for the moment. Let's just say beforehand that we are going to suppose some regularity of the initial datum φ, namely $\varphi \in H^m(\mathbb{R}^d)$ with $\frac{d}{2} < m \in \mathbb{N}_0$. This is particularly useful to deal with high values of γ and simplify many computations. Further comments on this assumption can be found in Chapter 4.

In Section 1.1, we study the properties of u and (NLS) from the beginning, to give a complete overview of the general theory and the tools we will use for the main result. In Section 1.2, we establish a strong space-time bound on u that will be crucial for establishing global existence, as well as for many other steps in this work.

1.1 Preliminary results

There is a strong and wide theory surrounding problem (NLS), thus we can have a lot of information about its solution u. We are now going to make a quick review of some basic notions and classical results. Essentially, what we need is that u exists globally in time and maintains the regularity of u_0. The results of this section are well known in the field; the reader may look at [12, 14] for an introduction to this topic, where almost all the results of this section can be found.

1.1.1 Classical local theory

First of all, we recall some definitions.

Definition 1.1.1. Given u a solution to (NLS), we define the rescaled solution u_λ, $\lambda > 0$, as

$$u_\lambda(t, x) = \lambda^{\frac{2}{1-\gamma}} u\left(\tfrac{t}{\lambda^2}, \tfrac{x}{\lambda}\right).$$

The action of the group of operators $\{\_\_{\lambda}\}_{\lambda\in\mathbb{R}^+}$ is often called *rescaling* and acts on the space of (local) solutions[1] of (NLS). The *critical regularity index* for the nonlinear Schrödinger equation is the number s_c given by

$$s_c = \chi^{-1}(\gamma) := \frac{n}{2} - \frac{2}{\gamma-1}, \qquad 1 < \gamma < \infty.$$

It is defined as the real number s such that the $\dot{H}^s$-norm of the rescaled solution u_λ at time 0

$$\||D|^s u_\lambda(0,\cdot)\|_{L^2(\mathbb{R}^n)} = \lambda^{\frac{n}{2}-\frac{2}{\gamma-1}-s}\||D|^s u(0,\cdot)\|_{L^2(\mathbb{R}^n)} \tag{1.2}$$

is a constant function of λ. The inverse function is given by

$$\chi(s) = 1 + \frac{4}{n-2s}, \qquad -\infty < s < \frac{n}{2}.$$

Finally, if the quantity (1.2) is a decreasing function of λ, the setting is called *subcritical*, or γ is said to be an *s-subcritical* power. The condition for this to happen is $\gamma < \chi(s)$ (i.e. $s > s_c$) if $s < \frac{n}{2}$, whereas there are no conditions if $s \geq \frac{n}{2}$. The remaining settings are called *supercritical*.

When having to specify the dimension, we will write $\chi_n(s)$.

Remark 1.1.2. Subcritical powers γ are expected to be well-behaving, because one would like to have longer solutions in time from small initial data. If $s = 1$, the critical powers are $1 < \gamma < 1 + \frac{4}{n-2}$ if $n \geq 3$, and $1 < \gamma < \infty$ for $n = 1, 2$. Note that if $s = 1$, then $1 + \frac{4}{n-2} = 2^* - 1$, and the 1-subcritical (and critical for $n \geq 3$) γ are exactly those for which it holds

$$\psi \in H^1(\mathbb{R}^n) \implies |\psi|^{\gamma-1}\psi \in H^{-1},$$

and equation (NLS) makes sense in H^{-1}. This is an easy consequence of Hölder's inequality and Sobolev embeddings.

[1] The scaling group acts on local classical C^2 solutions of (NLS) as well as on those given by Definition 1.1.3. For the latter, one must use the rescaling properties of the Fourier transform.

Definition 1.1.3. The initial value problem is said to be *locally well-posed* in $H^s(\mathbb{R}^n)$, $s \geq 0$ if the following conditions hold: there exists a unique $u \in X_T \hookrightarrow L^\infty([-T,T], H^s)$ which is a fixed point of the operator

$$Lu(t) = e^{it\Delta}u_0 - i\int_0^t e^{i(t-s)\Delta}|u(s)|^{\gamma-1}u(s)ds\,, \tag{1.3}$$

where X_T is a certain Banach space; also, $u \in C([-T,T], H^s)$, $0 < T = T(\|u_0\|_{H^s})$ is a non-increasing function of the norm of the initial datum, and the map $u_0 \mapsto u$ is continuous from $H^s \supset B_0(R)$ to $C([-T(R), T(R)], H^s)$. The initial value problem is *globally well-posed* if the above conditions hold for arbitrary $T > 0$. If the above definition is true for $X_T = L^\infty([-T,T], H^s)$, the well-posedness is said to be *unconditional* (see, e.g., [14, Def. 3.4]).

Remark 1.1.4. Actually, if T depends on u_0 but not on his H^s norm, the problem is often called well-posed *in the critical sense*, while the one we stated above is the local well-posedness *in the subcritical sense*.

If we have local well-posedness, it makes sense to talk about *the* maximal solution of the Cauchy problem, since two different solutions must coincide wherever their supports are not disjoint. It is defined on an open time interval $(-T^-, T^+)$, since a solution on a closed interval can be extended again for a certain time.

Remark 1.1.5. (Blowup criterion) If the IVP is locally well-posed (in the subcritical sense) in H^s, since the lifetime depends only on the norm of u_0, we have an immediate result: if u is the maximal solution on the time interval $(-T^-, T^+)$, then one of the following occurs:

$$T^\pm = +\infty \qquad \vee \qquad \lim_{t\to T^\pm}\|u(t)\|_{H^s} = +\infty. \tag{1.4}$$

It is immediate from the definition that the problem can't be well-posed in the supercritical case, at least once one can exhibit a solution that blows up in finite time. In fact, only through the rescaling action we could construct a succession of initial data converging to 0 in H^s such that the lifespan of the corresponding solutions goes to 0 as well. In general, the problem is expected to be at least locally well-posed in the subcritical case, ill-posed in the supercritical case and *locally well-posed in the critical sense* in the critical case.

The main theorem of this subsection is the following, which was first proved by Ginibre and Velo in [5] (see e.g. [12, §4], for a more complete result).

Theorem 1.1.6. *The initial value problem* (NLS) *is globally well-posed in* $H^1(\mathbb{R}^n)$ *if* γ *is* 1*-subcritical, i.e.* $1 < \gamma < \chi(1)$.

We will come back later to the globality statement of this theorem. Note that, since we have to take $n = 2$ in our original problem, for all $\gamma > 1$ we have local well-posedness in $H^1(\mathbb{R}^2)$. We actually have chosen the number 2 because the 1-critical power is ∞ and we will be able to work with arbitrary high powers γ.

Concerning general regularity assumptions, the pattern is the following: assuming enough regularity on the nonlinear term, local well-posedness in $H^s(\mathbb{R}^n)$ of (NLS) holds for $1 < \gamma < \chi(s)$, and in the critical case $\gamma = \chi(s)$ the problem is well-posed in the critical sense, namely the lifespan of the solution depends on a more complex quantity. Some works collecting the main results about this topic are [1], [2] and [7].

With regard to the cases we are interested in, unconditional local well-posedness holds in H^m for every $\gamma > m > \frac{n}{2}$. The proof can be found in [14, Prop. 3.8]; see also [1, §4.10][2]. In our case, this is true for $m \geq 2$. We obviously ask for $\gamma > m$ in order to have $|\cdot|^{\gamma-1}\cdot \in C^m(\mathbb{C}, \mathbb{C})$ and $|u|^{\gamma-1}u$ regular.

1.1.2 Globality and higher regularity

Our aim is to find global solutions for (1.1), so we need at least globality for u, that is the case $\varphi = 0$. In the defocusing case, global well-posedness in Theorem 1.1.6 follows from local well-posedness thanks to the conservation laws. In fact, the following quantities,

$$\|u(T)\|_{L^2} =: M(T) = M, \quad \|u(T)\|_{\dot{H}^1}^2 + \frac{1}{\gamma+1}\|u(T)\|_{L^{\gamma+1}}^{\gamma+1} =: E(T) = E\,,$$

[2]We will use a very similar technique in the proof of Proposition 2.2.1.

mass and energy, are (locally in time) conserved for sufficiently smooth and decaying solutions (and, by a density argument, for all H^1 solutions). In fact, for M, you can multiply the equation in (NLS) by $\bar{u}$, integrate in the whole space, take the imaginary part and obtain the result; you have to multiply by $\nabla \bar{u}$ instead to obtain conservation of E (this is a classical exercise). In particular, the $H^1(\mathbb{R}^n)$ norm is uniformly bounded. Since the time T only depends on the norm of the initial datum, the blowup criterion (1.4) ensures that the maximal solution is global in time. A simple iteration argument immediately shows globality too (it is essentially the same argument).

We now hope to obtain global solutions also in H^m, $\gamma > m \geq 2$. Notice that we already have a global solution $u \in C(\mathbb{R}, H^1)$, and if we take $u_0 \in H^m$, then $u \in C([-T,T], H^m)$ for a small time T. So we have to prove what is called a *persistence of regularity* result, i.e. that the solution u already existing belongs to $L^\infty_T H^m$ for arbitrary T. This is not trivial, as this time we don't have higher-order conserved energies and we don't know how to apply the blowup criterion.

There are several ways to prove persistence of regularity, without a priori estimates as in the H^1 case. First, it can be shown in the proof of Theorem 1.1.6 making an extra effort. Alternatively, we can use the *energy method* (see [14, §3.3]) as follows. We're only going to examine the case $n = 2$.

Lemma 1.1.7 (Moser Estimates; cf. [15, Prop. 3.7])**.** *Let $k \in \mathbb{N}_0$ and $f, g \in H^k(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Then, $fg \in H^k(\mathbb{R}^n)$ and*

$$\|fg\|_{H^k} \lesssim \|f\|_{H^k}\|g\|_{L^\infty} + \|f\|_{L^\infty}\|g\|_{H^k}. \tag{1.5}$$

Inductively, if $\mathbb{N} \ni p \geq 1$, then

$$\|f^p\|_{H^k} \lesssim \|f\|_{H^k}\|f\|_{L^\infty}^{p-1}. \tag{1.6}$$

In particular, if $k > \frac{n}{2}$, H^k is a quasi-Banach algebra, *i.e.*

$$\|fg\|_{H^k} \lesssim \|f\|_{H^k}\|g\|_{H^k}.$$

Proof. One just writes down the derivatives of fg and estimates them using Gagliardo–Nirenberg inequality, in particular

$$\|D^l f\|_{L^{\frac{2k}{l}}} \lesssim \|f\|_{L^\infty}^{1-l/k}\|D^k f\|_{L^2}^{l/k}.$$ □

Proposition 1.1.8 (Persistence of regularity, [14, Prop. 3.11])**.** *Let u be a local H^s solution to (NLS) in a time interval $I \ni 0$, with $\gamma > s$ or $\gamma \in 2\mathbb{N}_0+1$. Then, it holds*

$$\|u\|_{L^\infty_T H^s} \le \|u_0\|_{H^s} \exp\left(C_{n,s,\gamma} \|u\|^{\gamma-1}_{L^{\gamma-1}_T L^\infty}\right) \tag{1.7}$$

for each $T \in I$.

Proof. Assume $T > 0$ without loss of generality, thanks to the time reversal symmetry of (NLS). We do the standard energy estimate

$$\begin{aligned} \|u\|_{L^\infty_T H^s} &\le \|u_0\|_{H^s} + \int_0^T \||u(s)|^{\gamma-1}u(s)\|_{H^s} ds \le \\ &\le \|u_0\|_{H^s} + C_{n,m,\gamma} \int_0^T \|u(s)\|^{\gamma-1}_{L^\infty} \|u(s)\|_{H^s} ds\,, \end{aligned} \tag{1.8}$$

where we used Lemma 1.1.7 in the last inequality. From Gronwall's lemma, it follows that

$$\|u\|_{L^\infty_T H^m} \le \|u_0\|_{H^m} \exp\left(C_{n,m,\gamma} \|u\|^{\gamma-1}_{L^{\gamma-1}_T L^\infty}\right)$$

□

Remark 1.1.9. This proof only works if γ is an odd number and s is an integer, since we used inductively Lemma 1.1.7. It can be shown that a similar statement to Lemma 1.1.7 holds for real $p > k$, namely,

$$\||\phi|^{\gamma-1}\phi\|_{H^k} \lesssim \|\phi\|^{\gamma-1}_{L^\infty} \|\phi\|_{H^k} \tag{1.9}$$

for $k \in \mathbb{N}$, $k < \gamma \in \mathbb{R}$, which are the conditions we need in Proposition 1.1.10. Another case that is relatively simple is when γ is an odd number and k is replaced with $s \in [0, +\infty)$ (see [14, Lemma A.8]). For the general case of the inequality where $\gamma > s$ are real numbers and $\gamma > 1$, the reader may look at [11, §5.4.3].

Proposition 1.1.10. *Let $u \in C(\mathbb{R}, H^1(\mathbb{R}^2))$ be the global solution given by Theorem 1.1.6. Suppose that $u_0 \in H^s(\mathbb{R}^2)$, $s \in (1, +\infty)$, and either $s < \gamma$ or $\gamma \in 2\mathbb{N}_0 + 3$. Then, $u \in C(\mathbb{R}, H^s(\mathbb{R}^2))$ and the following estimate holds:*

$$\|u\|_{L^\infty_T H^s} \le \|u_0\|_{H^s} C_1 e^{C_2|T|}, \tag{1.10}$$

where all the constants explicitly written depend on n, s, γ, M, E.

Proof. As in the proof of Proposition 1.1.8, we assume $T > 0$. Clearly, we only have to prove the upper bound (1.10) to $\|u\|_{L^\infty_T H^s}$, and the globality in time follows from the blowup criterion. From conservation of mass and energy and Sobolev embeddings, it follows that

$$\|u\|_{L^\infty_T L^N} \lesssim_N \|u\|_{L^\infty_T H^1} \leq M + E^{\frac{1}{2}} \qquad \forall\, 2 \leq N < +\infty,\ \forall\, T > 0.$$

Using the Strichartz estimates. We immediately obtain, for all admissible pairs (p, q) such that $q > 2$, that

$$\|u\|_{L^p_T W^{1,q}} \lesssim \|u_0\|_{H^1} + \||u|^{\gamma-1}u\|_{L^1_T H^1}.$$

We can estimate the H^1 norm of the nonlinearity as below:

$$\||u|^{\gamma-1}u\|_{H^1} = \||u|^{\gamma-1}u\|_{L^2} + (\gamma-1)\||u|^{\gamma-1}\nabla u\|_{L^2} \lesssim \|u\|^{\gamma-1}_{L^N}\|u\|_{W^{1,q}},$$

where $N = \frac{2q}{q-2}(\gamma - 1)$, such that $2 \leq N < \infty$. Then we can continue:

$$\begin{aligned}\|u\|_{L^p_T W^{1,q}} &\lesssim \|u_0\|_{H^1} + \|u\|^{\gamma-1}_{L^\infty_T L^N}\|u\|_{L^1_T W^{1,q}} \lesssim \\ &\lesssim \|u_0\|_{H^1} + (M + E^{\frac{1}{2}})^{\gamma-1}T^\alpha \|u\|_{L^p_T W^{1,q}},\end{aligned}$$

with $\alpha = \frac{1}{p'}$. It follows that

$$\|u\|_{L^p_T W^{1,q}} \lesssim \|u_0\|_{H^1}, \qquad T \leq T_0 := (2C(M + E^{\frac{1}{2}})^{\gamma-1})^{-p'}.$$

Iterating the same argument in time, since u is unifromly bounded,

$$\|u\|_{L^p_T W^{1,q}} \lesssim \|u_0\|_{H^1}\left(\left\lceil \frac{T}{T_0}\right\rceil\right)^{\frac{1}{p}} \leq \|u_0\|_{H^1}\left(\frac{T}{T_0} + 1\right)^{\frac{1}{p}} \qquad \forall T > 0.$$

In other words,

$$\|u\|_{L^p_T W^{1,q}} = O_{M,E}(T^{\frac{1}{p}}) \quad \text{as } T \to +\infty.$$

Since $q > 2$, $W^{1,q}(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$, and since p ranges in $(2, \infty)$,

$$\|u\|_{L^p_T L^\infty} = O_{M,E}(T^{\frac{1}{p}}) \quad \text{as } T \to +\infty \tag{1.11}$$

holds for all $p > 2$ and using Hölder's inequality, for all $p \geq 1$. The statement then follows from (1.11) and Proposition 1.1.8. □

Remark 1.1.11. Inequalitiy (1.8) tells us easily that, if $m > \frac{n}{2}$, called u the $H^m(\mathbb{R}^n)$ solution to NLS, then $\|u(t)\|_{H^m}$ stays bounded for $t \to T$ if (and only if) $\|u(t)\|_{L^\infty}$ stays bounded. The 'only if' part is obvious since one norm controls the other; the 'if' follows from (1.8) and Gronwall's lemma. This tells us that an H^{m_1} solution is also an H^{m_2} solution for all times if $m_1 < m_2$ and the initial datum lies in H^{m_2}. Roughly speaking, persistence of regularity is easy if we work strictly above $\frac{n}{2}$. With some more techniques that we will use in Chapter 2, this statement happens to be true also if the nonlinearity is an arbitrary function $f \in C^m(\mathbb{C}, \mathbb{C})$ and $f(0) = 0$.

1.2 Decay in time

From now on, since equation (NLS) is symmetric with respect to time reflections, we study this equation in the time interval $[0, +\infty)$, and we suppose $T > 0$ in all the section. In the spirit of establishing globality, we would like to control the L^∞ norm of $\tilde{u}$ and find some decay to 0 for $t \to \infty$. We only have conservation of the H^1 norm, so we can't have a uniform $L^\infty L^\infty$ bound by a hair using only Sobolev embeddings. A first approach can be the following.

Corollary 1.2.1. *Let $u \in C(\mathbb{R}, H^1(\mathbb{R}^2))$ be the global solution to (NLS) with initial datum $u_0 \in H^2$ and $\gamma > m$. Then,*

$$\|u\|_{L^\infty_T L^\infty} \leq C_0\, O(\log\|u\|_{H^2} + C_1 + C_2 T)^{\frac{1}{2}} \qquad \text{as } T \to +\infty, \tag{1.12}$$

where all the constants explicitly written depend on n, m, γ, M, E.

The proof of this corollary follows from Theorem 1.1.10 and the following lemma.

Lemma 1.2.2. *Let $\psi \in H^2(\mathbb{R}^2)$. Then:*

$$\|\psi\|_{L^\infty} \leq \|\psi\|_{H^1} \tau\left(\log\left(\frac{\|\psi\|_{H^2}}{\|\psi\|_{H^1}}\right)\right). \tag{1.13}$$

where $\tau\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a universal non-decreasing C^1 function such that $\tau = O(T^{\frac{1}{2}})$ for $T \to +\infty$.

Proof. (without straightforward calculus) $H^s \hookrightarrow L^\infty$ for all $1 < s \in \mathbb{R}$. In particular,

$$|\psi(x)| = |\int e^{ix\xi}\widehat{\psi}(\xi)d\xi| \leq \int \langle\xi\rangle^{-s} \langle\xi\rangle^{s} |\widehat{\psi}(\xi)|d\xi \leq$$
$$\leq \|\langle\xi\rangle^{-s}\|_{L^2}\|\langle\xi\rangle^{s}\widehat{\psi}\|_{L^2} =: \widetilde{C}_s\|\psi\|_{H^s}.$$

With a short computation, we find out that

$$\widetilde{C}_s^2 = \int \frac{1}{(1+|\xi|^2)^s}d\xi = \frac{\pi}{(s-1)}.$$

Next, by complex interpolation

$$\|\psi\|_{H^s} \leq \|\psi\|_{H^1}^{(2-s)}\|\psi\|_{H^1}^{(s-1)}$$

for all $s \in [1,2]$. So, we have that, for all these s,

$$\|\psi\|_{L^\infty} \leq \sqrt{\frac{\pi}{(s-1)}}\|\psi\|_{H^1}^{(2-s)}\|\psi\|_{H^1}^{(s-1)} = \sqrt{\frac{\pi}{(s-1)}}\|\psi\|_{H^1}e^{\log(s-1)(\frac{\|\psi\|_{H^2}}{\|\psi\|_{H^1}})},$$

hence (1.13) follows for

$$\tau(T) := \sqrt{\pi}\inf_{s\in(0,1]}\left\{\frac{1}{\sqrt{s}}e^{sT}\right\}.$$

A standard study of this function leads to:

$$\tau(T) = \begin{cases} \sqrt{\pi}e^T & \text{for } T \leq \frac{1}{2}, \\ \sqrt{2\pi eT} & \text{for } T \geq \frac{1}{2}. \end{cases}$$

□

This estimate is quite simple to obtain, but it isn't optimal at all. In the linear case, we know that the $\mathbb{R}^n$ solution $u_{lin}(t) = e^{it\Delta}u_0$ is very strongly decaying, namely

$$\|u_{lin}(t)\|_{L^\infty} \lesssim |t|^{-\frac{n}{2}}\|u_0\|_{L^1},$$

and we could in principle expect something similar in the defocusing case (we surely can't expect much better, because equation (NLS) tends to have a linear behaviour for small values of u). To reach this kind of estimate, we are going to introduce the so-called *pseudo-conformal transform* (cf. [14, p.72]):

$$u(t,x) =: t^{-\frac{n}{2}} U\left(-\frac{1}{t}, \frac{x}{t}\right) e^{i\frac{|x|^2}{4t}}. \tag{1.14}$$

The map $u \mapsto U$ defined by (1.14) sends classical $C^1_t C^2_x$ solutions to (NLS) with time interval $I = [1, +\infty)$ into classical solutions of the equation

$$\begin{cases} iU_T + \Delta_X U = (-T)^\alpha |U|^{\gamma-1} U & \text{in } \widetilde{I} \times \mathbb{R}^n, \\ U(-1, X) = U_{-1}(X), \end{cases} \tag{1.15}$$

where $\alpha = \frac{n}{2}(\gamma - 1) - 2$, and

$$T := -\frac{1}{t}, \quad X := \frac{x}{t}, \quad \widetilde{I} := [-1, 0)\,.$$

Let's study the problem (1.15) for $n = 2$. The necessary assumption we make in order to avoid the singularity of the equation in $T = 0$ is $\gamma \geq 3$. With this assumption, $T \mapsto (-T)^{\frac{n}{2}(\gamma-1)-2}$ is smooth, bounded and positive in $\widetilde{I}$. Note that $3 = \chi_2(0)$: this is not a coincidence, since, for general n, the pseudo-conformal transform actually preserves the equation in the mass-critical case, making the term $(-T)^\alpha$ disappear.

The first remarkable fact is that H^1 local well-posedness can be proven in the same way as in problem (NLS), since the factor before the nonlinearity does not affect Strichartz estimates at all. Analogously, there are again conservation laws. In fact, the L^2 norm is still conserved, since the nonlinearity is just multiplied by a real constant (same calculations of the classical (NLS)). For the energy conservation we proceed as follows:

$$\begin{aligned} & iU_T + \Delta_X U = (-T)^\alpha |U|^{\gamma-1} U \\ \Longrightarrow\ & iU_T \overline{U}_T + \Delta_X U \overline{U}_T = (-T)^\alpha |U|^{\gamma-1} U \overline{U}_T \\ \overset{\int \mathrm{Re}}{\Longrightarrow}\ & -\frac{1}{2}\frac{d}{dt}\int |\nabla U|^2 = \frac{(-T)^\alpha}{\gamma+1}\frac{d}{dt}\int |U|^{\gamma+1} = \\ = & \frac{d}{dt}\left[\frac{(-T)^\alpha}{\gamma+1}\int |U|^{\gamma+1}\right] + \frac{\alpha}{\gamma+1}(-T)^{\alpha-1}\int |U|^{\gamma+1}. \end{aligned}$$

It follows, at least formally, that

$$E(T) := \|U(T)\|_{\dot{H}^1} + \frac{(-T)^\alpha}{\gamma+1}\|U(T)\|_{L^{\gamma+1}} \leq E(-1) =: E.$$

Using conservation of mass and energy, we easily establish global well-posedness (in the time interval $\widetilde{I}$) in H^1 and persistence of regularity in the same way as we did for u. In short, Theorem 1.1.6 and Proposition 1.1.10 hold for the solution U too in the time interval $\widetilde{I}$.

If $U_{-1} \in H^m(\mathbb{R}^2)$, $m \geq 2$, there is a solution U to (1.15) in $C_{\widetilde{I}}H^m$ and, thanks to the estimate (1.10),

$$\|U\|_{L^\infty_{\widetilde{I}}H^m} \leq \|U_{-1}\|_{H^2} C_{m,\gamma,M,E},$$

Thus, taking $m = 2$, by Sobolev embedding,

$$\|U\|_{L^\infty(\widetilde{I}\times\mathbb{R}^2)} \leq \|U_{-1}\|_{H^2} C_{\gamma,M,E}. \tag{1.16}$$

We can see the above pointwise estimate in terms of the original solution u and conclude with the following proposition.

Proposition 1.2.3. *Let $u_0 \in H^{2,2}(\mathbb{R}^2)$. Then, the solution to (NLS) with $n = 2$ and $\gamma \geq 3$ satisfies*

$$\|u(t)\|_{L^\infty} \leq \|u_0\|_{H^{2,2}} C_{\gamma,M,E} |1+t|^{-1} \qquad \forall\, t > 0. \tag{1.17}$$

Definition 1.2.4 ([14, §2.4])**.** The *weighted Sobolev space* $H^{k,k}(\mathbb{R}^n)$ is the closure of the Schwartz functions under the norm

$$\|\psi\|_{H^{k,k}} := \sum_{i=0}^{k} \|\langle x\rangle^i \psi\|_{H^{k-i}}.$$

Proof of Proposition 1.2.3. The proof follows from the above arguments, taking the translation $u(t-1)$ (to set the initial time $t = 1$ and avoid singularities) and then using the pseudo-conformal transform. In fact, we have that $u_0 \in H^2$ and through simple calculations,

$$u_0 \in H^{2,2} \implies u_0 e^{-i\frac{|x|^2}{4t}} \in H^2 \implies U_{-1} \in H^2.$$

This means that the solution U of (1.15) with initial data is defined and bounded in $[0, 1)$ thanks to (1.16). Inverse pseudo-conformal transforming[3] brings us to the thesis. □

[3] Actually, we have to prove that (1.14) maps H^2 solutions into H^2 solutions. This transition between u and U is made rigorous using a standard density argument and the local well-posedness of problems (NLS) and (1.15). We shall omit the details.

The above result states that for regular and smooth functions, we have a good decay in time. We have made the first step towards the solution. We could ask ourselves what can we say about higher order derivatives. We can use again persistence of regularity.

Corollary 1.2.5. *In the hypotheses of Proposition 1.2.3, if* $u_0 \in H^s$*,* $s < \gamma$ *or* $\gamma \in 2\mathbb{N}_0 + 3$*, we also have:*

$$\|u\|_{L^\infty_T H^s} \lesssim_{u_0,\gamma} 1 \qquad \forall T \in \mathbb{R}.$$

Proof. Thanks to the decay, we immediately obtain that $\|u\|_{L^{\gamma-1}_{\mathbb{R}} L^\infty} < \infty$. Thus, the proof follows thanks to Proposition 1.1.8. □

Remark 1.2.6. Another consequence, thanks to Gagliardo–Nirenberg inequality (Proposition A.2.2), is a decay for the derivatives of u. In fact, if s is large enough, a fixed derivative of u decays in time with a power that is arbitrarily close to -1 from above.

Chapter 2

Local well-posedness of the main problem

In this chapter we come back to problem (1.1). Since we want to study the solution v as a perturbation around the solution $\widetilde{u}$, our new observable will be $w \coloneqq v - \widetilde{u}$. Subtracting the two equations (1.1) and (NLS), we obtain the equation for the perturbation w (we rename φ as w_0 to make the notation consistent):

$$\begin{cases} iw_t + \Delta w = g[w] & \text{in } \mathbb{R}^+ \times \mathbb{R}^d, \\ w(0, \mathbf{x}) = w_0(\mathbf{x}), \end{cases} \tag{NLSP}$$

where $g[w] = f(w + \widetilde{u}) - f(\widetilde{u})$ and $f(u) = |u|^{\gamma-1}u$. As we can see, g depends on $\mathbf{x}$ and t, although this dependence is controlled thanks to (1.2.5). We are looking for local well-posedness in H^m, $m > d/2$. Since we are interested to work with high powers γ, i.e. $\chi^{-1}(\gamma)$ is close to $d/2$, this is a reasonable assumption to start with, even if not the most general.

In order to prove existence and uniqueness, we will need estimates for the map g, in order to have $g \colon H^m \to H^m$ and settle the contraction argument. These are obtained in Section 2.1. Despite the fact that we will need the regularity assumption $\gamma \geq m + 1$, we will manage to obtain optimal estimates with respect to the regularity of the initial datum u_0.

2.1 Some estimates for the forcing term

Proposition 2.1.1. *Suppose $d \geq 3$, $f \in C^{m,1}_{loc}(\mathbb{C}, \mathbb{C})$, $m > d/2$ as a real function, i.e. f has m^{th} derivatives that are locally Lipschitz. Then,*

$$\|g[\psi]\|_{H^m(\mathbb{R}^d)} \leq C_1(\|u\|_{H^m(\mathbb{R}^2)}, M)\|\psi\|_{H^m(\mathbb{R}^d)}, \tag{2.1}$$

$$\|g[\phi] - g[\psi]\|_{H^m(\mathbb{R}^d)} \leq C_1(\|u\|_{H^m(\mathbb{R}^2)}, M)\|\phi - \psi\|_{H^m(\mathbb{R}^d)}, \tag{2.2}$$

$$\|g[\psi]\|_{H^m(\mathbb{R}^d)} \leq C_2(\|u\|_{W^{m,\infty}(\mathbb{R}^2)}, M)\|\psi\|_{H^m(\mathbb{R}^d)}, \tag{2.3}$$

$$\|g[\phi] - g[\psi]\|_{H^m(\mathbb{R}^d)} \leq C_2(\|u\|_{W^{m,\infty}(\mathbb{R}^2)}, M)\|\phi - \psi\|_{H^m(\mathbb{R}^d)}, \tag{2.4}$$

$$\|g[\phi] - g[\psi]\|_{L^2(\mathbb{R}^d)} \leq C_3(\|u\|_{L^\infty(\mathbb{R}^2)}, M)\|\phi - \psi\|_{L^2(\mathbb{R}^d)}, \tag{2.5}$$

in the hypotheses

$$\|\phi\|_{L^\infty(\mathbb{R}^d)}, \|\psi\|_{L^\infty(\mathbb{R}^d)} \leq M \qquad \text{in (2.3),(2.5)}, \tag{2.6}$$

$$\|\phi\|_{H^m(\mathbb{R}^d)}, \|\psi\|_{H^m(\mathbb{R}^d)} \leq M \qquad \text{in (2.1),(2.2),(2.4)}. \tag{2.7}$$

Estimates (2.3) and (2.4) are more 'rough'. Their proof is easier, but we have to assume a higher regularity of the initial datum u_0 in order to use them for our problem. Nonetheless, they are stronger in the sense that the $W^{m,\infty}$ norm goes to zero as $t \to +\infty$ in some hypotheses (see Remark 1.2.6). Estimates (2.1) and (2.2) are refined versions, which are proved studying the flexibility of the $H^m(\mathbb{R}^d)$ norm in relation to the non-isotropy of the problem.

The outline for the proof is taken from [1, Lemma 4.10.2], but we follow the original idea to consider mixed norms with respect to the variables x and z in order to minimize the assumptions.

Proof. We shall omit the measure space if that is $\mathbb{R}^d$ or $\mathbb{R}^2$, if there's no risk of misunderstandings. We define, for $R \geq 0$,

$$K^i(R) := \sum_{|\beta|=i} \|D^\beta f\|_{L^\infty(\overline{B_0(R)})}, \qquad 1 \leq i \leq m+1,$$

where f is considered as a function in $C^{m,1}_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$. In the case $i = m + 1$, the terms of the sum become the Lipschitz constants on the ball instead (or we could say that we consider the $(m + 1)^{\text{th}}$ distributional derivative, that is the same thanks to the Rademacher theorem). We can easily see from Lagrange theorem that

$$|g[\psi](\mathbf{x})| \leq K^1(M + \|u\|_{L^\infty(\mathbb{R}^2)})|\psi(\mathbf{x})|\,, \tag{2.8}$$

$$|g[\phi](\mathbf{x}) - g[\psi](\mathbf{x})| \leq K^1(M + \|u\|_{L^\infty(\mathbb{R}^2)})|\psi(\mathbf{x}) - \phi(\mathbf{x})|\,, \tag{2.9}$$

thus we immediately obtain (2.5) taking the L^2 norm in (2.9) and we estimate $\|g[\psi]\|_{L^2}$ with the right sides of (2.1) and (2.3), in the hypothesis (2.6) (as well as (2.7)).

To show (2.2), we only need to control the derivative

$$\|D^\alpha(g[\phi] - g[\psi])\|_{L^2}$$

for every $|\alpha| = m$ thanks to Gagliardo–Nirenberg inequality.
The term $D^\alpha\left(g[\phi] - g[\psi]\right) = D^\alpha f(\phi + \tilde{u}) - D^\alpha f(\psi + \tilde{u})$, using m times the chain rule, is a sum of terms of the form

$$D_{\mu_1\ldots\mu_k} f(\phi+\tilde{u}) \prod_{j=1}^{k} D^{\beta_j}(\phi+\tilde{u})_{\mu_j} - D_{\mu_1\ldots\mu_k} f(\psi+\tilde{u}) \prod_{j=1}^{k} D^{\beta_j}(\psi+\tilde{u})_{\mu_j}\,, \tag{2.10}$$

where $1 \leq k \leq m$, $\sum \beta_j = \alpha$ and μ_j are indices corresponding to real and imaginary components of the functions. This term can in turn be written as a sum of terms, where the first one is

$$\Big[D_{\mu_1\ldots\mu_k} f(\phi+\tilde{u}) - D_{\mu_1\ldots\mu_k} f(\psi+\tilde{u})\Big] \prod_{j=1}^{k} D^{\beta_j}(\phi+\tilde{u})_{\mu_j}\,,$$

and the other ones are of the following form:

$$D_{\mu_1\ldots\mu_k} f(\psi+\tilde{u}) \prod_{j=1}^{k} D^{\beta_j} \zeta^{(j)}_{\mu_j}\,,$$

where $\zeta^{(j)}$ are either $\psi + \tilde{u}$ or $\phi + \tilde{u}$, but at least one of them is $\phi - \psi$.
Finally, by a further decomposition, we see that one just has to estimate the following two terms, the first one being

$$\Big[D_{\mu_1\ldots\mu_k} f(\phi+\tilde{u}) - D_{\mu_1\ldots\mu_k} f(\psi+\tilde{u})\Big] \prod_{j=1}^{k} D^{\beta_j} \eta^{(j)}_{\mu_j}, \tag{2.11}$$

and the other one being

$$D_{\mu_1\dots\mu_k} f(\psi+\widetilde{u}) \prod_{j=1}^{k} D^{\beta_j}\zeta^{(j)}_{\mu_j}, \tag{2.12}$$

where $\eta^{(j)}$ are either ϕ or $\widetilde{u}$ without any restriction, and $\zeta^{(j)}$ can be ϕ, ψ and $\widetilde{u}$, but at least one of them is $\phi-\psi$.

Remark (a)**.** If we consider mixed norm spaces on a product $\mathbb{R}^m \times \mathbb{R}^n$ (or even general measure spaces), we have as a direct consequence of integral Minkowski inequality,

$$L^p_{\mathbb{R}^m} L^q_{\mathbb{R}^n} \hookrightarrow L^q_{\mathbb{R}^n} L^p_{\mathbb{R}^m},$$

with exact inequality between the norms, whenever $p \le q$.

Remark (b)**.** We have $H^m(\mathbb{R}^d) \hookrightarrow L^\infty_x L^2_z$. In fact,

$$\begin{aligned} \psi \in H^m \implies D^\alpha \psi \in L^2_z L^2_x \quad \forall |\alpha| \le m \\ \implies \| \|\psi\|_{L^2_x} + \|\nabla_x \psi\|_{L^2_x} + \dots + \|D^m_x \psi\|_{L^2_x} \|_{L^2_z} < \infty \\ \implies \psi \in L^2_z H^m_x \overset{m\ge 2}{\hookrightarrow} L^2_z L^\infty_x \hookrightarrow L^\infty_x L^2_z. \end{aligned}$$

Now, let's call $D^\gamma := D_{\mu_1\dots\mu_k}$ for the sake of brevity. Thanks to (2.9), taking $D^\gamma f$ instead of f, we have

$$\begin{aligned} \|D^\gamma f(\phi+\widetilde{u}) - D^\gamma f(\psi+\widetilde{u})\|_{L^\infty_x L^2_z} \lesssim \\ \lesssim K^{k+1}(M + \|u\|_{L^\infty(\mathbb{R}^2)}) \|\phi-\psi\|_{L^\infty_x L^2_z} \lesssim \\ \lesssim K^{k+1}(M + \|u\|_{L^\infty(\mathbb{R}^2)}) \|\phi-\psi\|_{H^m}. \end{aligned} \tag{2.13}$$

In a similar way, we have

$$\|D^\gamma f(\phi+\widetilde{u}) - D^\gamma f(\psi-\widetilde{u})\|_{L^p} \lesssim K^{k+1}(M + \|u\|_{L^\infty(\mathbb{R}^2)}) \|\phi-\psi\|_{L^p} \tag{2.14}$$

for every $1 \le p \le +\infty$.

Remark (c)**.** Set $p_j := \frac{2m}{|\beta|_j}$. From Gagliardo–Nirenberg inequality,

$$\|D^{\beta_j}\varphi\|_{L^{p_j}(\mathbb{R}^n)} \lesssim \|\varphi\|^{\frac{|\beta|_j}{m}}_{H^m(\mathbb{R}^n)} \|\varphi\|^{1-\frac{|\beta|_j}{m}}_{L^\infty(\mathbb{R}^n)} \quad \Big(\overset{m>\frac{n}{2}}{\lesssim} \|\varphi\|_{H^m(\mathbb{R}^n)} \Big). \tag{2.15}$$

From that,

$$\|D^{\beta_j}\tilde{u}\|_{L_x^{p_j}L_z^\infty} \lesssim \|u\|_{H^m(\mathbb{R}^2)}$$
$$\|D^{\beta_j}\psi\|_{L^{p_j}} \lesssim \|\psi\|_{H^m(\mathbb{R}^2)} \implies D^{\beta_j}\psi \in L_z^{p_j}L_x^{p_j},$$

and, from Remark (b),

$$\psi \in L_z^2 H_x^m \implies D^{\beta_j}\psi \in (L_z^2 \cap L_z^{p_j})L_x^{p_j}.$$

In particular,

$$D^{\beta_j}\psi \in L_z^{q_j}L_x^{p_j} \hookrightarrow L_x^{p_j}L_z^{q_j}$$

for every $q_j \in [2, p_j]$ (the same holds for ϕ and $\phi - \psi$).

We are ready to estimate (2.11) and (2.12). Remember that $\sum_{j=1}^k \frac{1}{p_j} = \frac{1}{2}$. We first decompose the norm of the derivatives in the product of the norms of their terms unsin Hölder's inequality.

$$\|(2.11)\|_{L^2} \lesssim \|D^\gamma f(\phi+\tilde{u}) - D^\gamma f(\psi+\tilde{u})\|_{L^\infty} \prod_{j=1}^k \|D^{\beta_j}\eta^{(j)}\|_{L_x^{p_j}L_z^{q_j}},$$
$$\|(2.12)\|_{L^2} \lesssim \|D^\gamma f(\psi+\tilde{u})\|_{L^\infty} \prod_{j=1}^k \|D^{\beta_j}\zeta^{(j)}\|_{L_x^{p_j}L_z^{q_j}}.$$

We then make the following choices: we set in both the estimates p_j as above and q_j such that $q_j = \infty$ if the corresponding term $\eta^{(j)}$ or $\zeta^{(j)}$ is equal to $\tilde{u}$, whereas $q_j \in [2, p_j]$ if the corresponding term is ϕ, ψ or $\phi - \psi$. We also choose the q_j such that $\sum_{j=1}^k \frac{1}{q_j} = \frac{1}{2}$. This can always be done for the term (2.12), because $\zeta^{(j)} = \phi - \psi$ for at least one index j. Concerning the term (2.11), the only case in which this estimate doesn't work is when $\eta^{(j)} = \tilde{u}\ \forall j$. In this case, we use the following alternative estimate:

$$\|(2.11)\|_{L^2} \lesssim \|D^\gamma f(\phi+\tilde{u}) - D^\gamma f(\psi+\tilde{u})\|_{L_x^\infty L_z^2} \prod_{j=1}^k \|D^{\beta_j}\tilde{u}\|_{L_x^{p_j}L_z^\infty}. \tag{2.16}$$

In all the cases, we can hence use the previous remarks to estimate all the pieces according to their structures:

$$\|(2.11)\|_{L^2} \overset{\substack{(2.13),(c)\\(2.14)}}{\lesssim} K^{k+1}(M + \|u\|_{L^\infty(\mathbb{R}^2)})\|u\|_{H^m}^h \|\phi\|_{H^m}^l \|\phi - \psi\|_{H^m},$$

$$\|(2.12)\|_{L^2} \overset{(2.14),(c)}{\lesssim} K^k(M + \|u\|_{L^\infty(\mathbb{R}^2)})\|u\|_{H^m}^r\|\phi\|_{H^m}^s\|\psi\|_{H^m}^t\|\phi-\psi\|_{H^m}$$

for some $h, l, r, s, t \in \mathbb{N}_0$ representing the number of $\widetilde{u}$, ϕ and ψ in the products, such that $h+l = r+s+t+1 = k$. Combining these estimates gives us (2.2). Furthermore, since $g[0] \equiv 0$, (2.1) immediately follows from (2.2) in the case $\psi = 0$.

Estimate (2.4) is actually much simpler: it is enough to completely forget about mixed norms and substitute the $L_x^{p_j}L_z^{q_j}$ norms with L^{q_j} norms everywhere, and the $L_x^{p_j}L_z^\infty$ norm with L^∞ norm in (2.16).
Again, supposing e.g. $\phi = 0$ (this choice is necessary to conclude easily the following argument), we automatically find (2.3). Nonetheless, in (2.3) we can suppose (2.6) instead of (2.7). This can be shown writing again the estimates we made. In (2.11) the surviving terms are those such that $\eta^{(j)} = \widetilde{u}\ \forall j$, so we can estimate as we just said:

$$\begin{aligned}\|(2.11)\|_{L^2} &\lesssim \|D^\gamma f(\psi+\widetilde{u}) - D^\gamma f(\widetilde{u})\|_{L^2}\prod_{j=1}^k\|D^{\beta_j}\widetilde{u}\|_{L^\infty} \overset{(2.14)}{\lesssim} \\ &\lesssim K^{k+1}(M+\|u\|_{L^\infty(\mathbb{R}^2)})\|u\|_{W^{m,\infty}}^k\|\psi\|_{L^2}.\end{aligned}$$

We then estimate (2.12) in the following way:

$$\begin{aligned}\|(2.12)\|_{L^2} &\lesssim \|D^\gamma f(\psi+\widetilde{u})\|_{L^\infty}\prod_{j=1}^k\|D^{\beta_j}\zeta^{(j)}\|_{L^{q_j}} \lesssim \\ &\overset{(c)}{\lesssim} K^k(M+\|u\|_{L^\infty(\mathbb{R}^2)})\|u\|_{W^{m,\infty}}^r\|\psi\|_{L^\infty}^s\|\psi\|_{H^m}.\end{aligned}$$

The last inequality is true if the q_j are chosen as above with a further condition: assuming for simplicity that $\zeta^{(j)} = \psi$ exactly for $1 \le j \le s+1$, calling $\sum_{j=1}^{s+1}|\beta_j| = b$, we set $q_j = \frac{2b}{|\beta_j|}$. In fact, consequently one has

$$\prod_{j=1}^{s+1}\|D^{\beta_j}\psi\|_{L^{q_j}} \lesssim \|\psi\|_{L^\infty}^s\|\psi\|_{H^b},$$

thanks to (2.15). This concludes the proof. □

Remark 2.1.2. Unfortunately, it is not enough to suppose $f \in C^m(\mathbb{C}, \mathbb{C})$. This means that in the local theorem we will have to suppose that $\gamma \ge m+1$.

This fact seems to be inevitable, because of the term where $\eta^{(j)} = \tilde{u} \quad \forall j$. In (2.16), we see that to the term $D^\gamma f(\phi + \tilde{u}) - D^\gamma f(\psi + \tilde{u})$ must be assigned an L^2 norm in the z directions, where $\tilde{u}$ and its derivatives are constant. That norm can't be controlled in any way by the H^m norm of $\phi - \psi$. In our case, where f is a pure power, we would need an L^p norm with $p < 2$, or even $p < 1$, in order to control the mass at infinity.

Remark 2.1.3. It is not clear whether or not we can assume (2.6) instead of (2.7) in (2.1). Using the same argument as in (2.3), it turns out that we would need a version of Gagliardo–Nirenberg inequality with mixed norms. That is beyond the scope of this work.

Some information on the constants C_1, C_2, C_3 will be useful. As one can check reading the previous proof, we can deduce a very elegant estimate.

Lemma 2.1.4. *Let $f(z) = |z|^{\gamma-1}z$, $z \in \mathbb{C}$. Assume that either $\gamma \geq m+1$, $m > \frac{d}{2}$, or $\gamma \in 2\mathbb{N}_0 + 3$. Then, we have the bounds*

$$C_i(h, M) \lesssim_{d,m} (h+M)^{\gamma-1}, \qquad i = 1, 2, 3.$$

Proof. One can check that for such a function f,

$$K^i(R) \lesssim_{d,i} R^{\gamma-i}\,, \qquad 1 \leq i \leq m+1\,,$$

and they are identically 0 for $i > \gamma$ if $\gamma \in 2\mathbb{N}_0 + 3$. The proof hence follows putting together all the terms appearing in the previous proof. $\square$

2.2 Local well-posedness

From the estimates of Proposition 2.1.1, we can obtain local existence in a standard way.

Theorem 2.2.1. *Fix $u_0 \in H^m(\mathbb{R}^2)$, $m > d/2$. Then, the problem (NLSP) is unconditionally locally well-posed in H^m if either $\gamma \geq m+1$, or $\gamma \in 2\mathbb{N}_0 + 3$.*

Proof. The proof follows a standard contraction argument. Let us call $X \coloneqq L^\infty_T H^m$, and $B_R \coloneqq \overline{B}_0(R),\ R > 0$. Consider the operator (1.3),

$$L[v](t) = e^{it\Delta} w_0 - i \int_0^t e^{i(t-s)\Delta} g[v(s)] ds\,.$$

We will call $h := \|u\|_{L^\infty_{\mathbb{R}} H^m(\mathbb{R}^2)}$, which is luckily finite thanks to Corollary 1.2.5. Using Proposition 2.1.1, we have for every $\|v_1\|_X, \|v_2\|_X \leq R$ that

$$\|L[v_1]\|_X \leq \|w_0\|_{H^m} + \|\int_0^t e^{i(T-s)\Delta} g[v_1(s)]ds\|_X \leq \tag{2.17}$$
$$\leq \|w_0\|_{H^m} + T\|e^{i(T-s)\Delta} g[v_1(s)]ds\|_{H^m} \leq \|w_0\|_{H^m} + T\|g[v_1]\|_X \leq$$
$$\leq \|w_0\|_{H^m} + TC_1(h,R)\|v_1\|_X.$$

Similarly, we have an estimate for the difference of two images of the operator:

$$\|L[v_1] - L[v_2]\|_X \leq TC_1(h,R)\|v_1 - v_2\|_X.$$

We can therefore choose R (first) and T (which depends on R; $T = T(R)$) such that

$$R \geq 2\|w_0\|_{H^m}, \qquad TC_1(h,R) \leq \frac{1}{2}, \tag{2.18}$$

and see that $L\colon B_R \to B_R$ is a contraction of a complete metric space, with Lipschitz constant $\frac{1}{2}$. So, we have a fixed point w of L, i.e. a solution of (NLSP) with initial datum w_0. Note that, for every initial datum w_0, a solution $w \in X$ that is a fixed point of the operator $L[\cdot]$ automatically satisfies $w \in C([0,T], H^m)$. In fact, the map

$$F \mapsto \int_0^t e^{i(t-s)\Delta} g[v(s)]ds$$

is continuous from $L^{\widetilde{p}'} L^{\widetilde{q}'}$ to $L^p L^q$ for every pair of Strichartz couples (p,q) $(\widetilde{p}', \widetilde{q}')$. Hence, continuity follows by a density argument.

Uniqueness in the ball B_R follows from the fixed point theorem. To prove uniqueness in the whole space X, we use a bootstrap argument (cf. [14], proposition 3.8). Let's consider $T = T(2R) \leq T(R)$ instead. Clearly, the unique solution given by the same argument in the ball B_{2R} must be w restricted to the new time interval. Assume if possible that there is another solution w^*. Then, one just needs to prove e.g. $\|w^*(t)\|_{H^m} \leq R$ thanks to uniqueness. However, we can make the bootstrap assumption $\|w^*(t)\|_{H^m} \leq 2R$, which implies the previous condition thanks to uniqueness in B_{2R}.

It remains to prove the continuous dependence on the initial datum. Given two functions v_0, w_0,

$$\|v_0\|_{H^m}, \|w_0\|_{H^m} \leq \frac{R}{2},$$

let's call v and w the corresponding solutions on the interval $[0, T]$. We have, in the same way as above,

$$\|v - w\|_X \leq \|v_0 - w_0\|_{H^m} + TC_1(h, R)\|v - w\|_X.$$

Subtracting the last term on both sides, and using the second condition in (2.18), we obtain

$$\|v - w\|_X \leq 2\|v_0 - w_0\|_{H^m}.$$

This tells us that the map

$$\begin{aligned} \mathcal{B}_{\frac{R}{2}} &\to X \\ v_0 &\mapsto v = L[v] \end{aligned}$$

is Lipschitz continuous with arbitrary R and T small enough. Here we wrote $\mathcal{B}_r := \overline{B}_r(0) \subset H^m$. □

Clearly, the proof still works if we use estimates (2.3), (2.4) instead of (2.1), (2.2).

Remark 2.2.2. In order to prove local existence and uniqueness, estimate (2.2) (or (2.4)) is not necessary. One could also equip B_R wih the distance induced by the norm of the space $L^\infty L^2$. With this topology, B_R is still a complete metric space.

> *Proof.* In fact, let $\{u_n\} \in B_R$, $u_n \to u$ in $L^\infty_T L^2$. It follows that $u_n(t) \to u(t)$ in L^2 for a.e. $t \in [0, T]$. But we also have $\|u_n(t)\|_{H^m} \leq R$ for a.e. $t \in [0, T]$ and all $n \in \mathbb{N}$; so, for a.e. t there exists a subsequence n_j such that $u_{n_j} \rightharpoonup v(t)$ in H^m, in particular $\|v(t)\|_{H^m} \leq \liminf \|u_{n_j}\|_{H^m}$. It follows that $u(t) \in H^m$ for a.e. t and $\|u(t)\|_{H^m} \leq R$ a.e. t, i.e. $u \in B_R$. □

Once we know that, we can use estimate (2.5) instead of (2.2) and settle the fixed point argument with the new metric (see [1]). This argument is quite useful, but we didn't need it since estimates (2.1) and (2.2) are proved with the same effort, at least in this particular setting.

In order to establish global existence, we would hope an intermediate result. Namely, we hope that with small initial data, the maximal solution lasts for an arbitrarily long time. This is true, since the nonlinearity is regular around 0, so it has a linear behaviour for small values of w.

Corollary 2.2.3. *(Lifespan) Let's call w the maximal solution of* (NLSP) *given in Theorem 2.2.1 on the interval* $[0, T^+)$*. Then,*

$$T^+ \geq \Theta(\|w_0\|_{H^m}),$$

where $0 < \Theta$ *is a non-increasing function, depending only on* d, m, γ, u_0*, such that*

$$\lim_{r \to 0} \Theta(r) = +\infty\,.$$

(cf. [7], theorem 4.2)

Proof. Let w be the maximal solution with initial datum w_0. Let's call

$$c(t) := \|w\|_{L^\infty_t H^m}, \qquad c_0 := c(0) = \|w_0\|_{H^m}.$$

We can rewrite inequality (2.17), using Lemma 2.1.4, as

$$c(t) = c_0 + \int_0^t C_1(h, c(s))c(s)\,ds \leq c_0 + C\int_0^t (h + c(s))^{\gamma-1}c(s)\,ds. \tag{2.19}$$

Using a bootstrap argument, one can estimate $c(t)$ with $x(t)$ for the times t in which it is defined, where x satisfies (2.19) with equality, i.e.

$$x(t) \leq c_0 + \int_0^t C(h + x(s))^{\gamma-1}x(s)\,ds.$$

The function x is thus a solution of

$$x'(t) = C(h + x(t))^{\gamma-1}x(t),$$

and when it comes to solve the ODE, we find

$$\int_{c_0}^{x(t)} \frac{dx}{C(h + x)^{\gamma-1}x} = t\,.$$

We see that the above integral converges for large values of x, while diverges as $c_0 \to 0$. This means that, calling

$$\Theta(r) := \int_r^{+\infty} \frac{dx}{C(h + x)^{\gamma-1}x}, \tag{2.20}$$

the solution w is bounded by a continuous function between times $t = 0$ and $t = \Theta(c_0)$, thus it can't blow up before that time.
The corollary is then proved, since the integral diverges logarithmically for r approaching zero. $\square$

With the same proof, we have the following lemma.

Lemma 2.2.4. *Let $0 < t_1 \in \mathbb{R}$ and $\varepsilon > 0$. Then, the solution to (NLSP) with initial datum w_0 exists on the whole interval $I = [0, t_1]$ and it is such that*

$$\|w\|_{L^\infty H^m} < \varepsilon \quad \text{if} \quad \|w_0\|_{H^m} < \delta\,,$$

where δ is such that

$$\Theta(\delta) - \Theta(\varepsilon) \geq t_1\,,$$

and where Θ is the function defined in (2.20).

The local theory we have developed so far works well for our problem, and the assumption $m > d/2$ is comfortable as we don't have to worry about how large the power γ can get. Although, the assumptions are far from being optimal. In Chapter 3 we will settle the argument for the global well-posedness of problem (NLSP), which is designed to hold with lower regularity assumptions; namely, it works in general in the subcritical regime $m < \chi^{-1}(\gamma)$, so that m can go below $d/2$. In Chapter 4, we make some comments on a possible extension of local theory to the subcritical case.

Chapter 3

Global Existence

Below we state the main theorem of this work. This chapter is dedicated entirely to its proof.

Theorem 3.0.1. *Assume* $m \in \mathbb{N}$*,* $m > d/2$*.*
a) Let $u_0 \in H^m(\mathbb{R}^2)$*. Let* $\gamma \geq m+1$ *or* $\gamma \in 2\mathbb{N}_0 + 3$*. The Cauchy problem* (NLSP) *with respect to the initial datum* w_0 *is then unconditionally locally well-posed in* $H^m(\mathbb{R}^d)$*.*
b) Furthermore, if $\gamma > m+2$ *or* $\gamma \in 2\mathbb{N}_0 + 3$*, and if* $u_0 \in H^{2,2} \cap H^s$*, where* $s > m+1$ *if* $\gamma > 3$ *and* $s > \lfloor 4m/3 \rfloor + 2$ *if* $\gamma = 3$*, then there exists* $\varepsilon = \varepsilon(u_0)$ *such that there exists a global solution of* (NLSP) *whenever* $\|w_0\|_{H^m} < \varepsilon$*.*

Part (a) was proved in Chapter 2. To prove part (b), we will need the decay of the solution u. As we can see, g is linear around $w = 0$, condition that would normally prevent global existence for small data if $\gamma > 1$. However, since u is small for large times, this linear behaviour will tend to vanish rapidly enough to be insignificant thanks to the time decay of u.

3.1 Linearization

A standard method to prove global existence for small data. Assume $\gamma \geq 3$. Let v be the solution to (NLS) with initial datum v_0. Let $\widehat{\gamma}$ be such that $(\gamma - 1, \widehat{\gamma})$ is a Strichartz pair, i.e. $\widehat{\gamma} = \frac{2(\gamma-1)n}{(\gamma-1)n-4}$ $\frac{n}{\gamma} < k \leq \gamma$. Note that $\frac{n}{\gamma}$

is exactly $\chi^{-1}(\gamma)$, as defined in Definition 1.1.1. We then call

$$M(T) := \|v\|_{L^\infty_T H^k} + \|v\|_{L^{\gamma-1}_T W^{k,\widehat{\gamma}}}.$$

Using Strichartz estimates, Lemma 1.9, Hölder's inequality and Sobolev immersions,

$$\begin{aligned} M(T) &\lesssim \|v_0\|_{H^k} + \||v|^{\gamma-1}v\|_{L^1_T H^k} \lesssim \\ &\lesssim \|v_0\|_{H^k} + \|v\|^{\gamma-1}_{L^{\gamma-1}_T L^\infty}\|v\|_{L^\infty_T H^k} \lesssim \\ &\lesssim \|v_0\|_{H^k} + \|v\|^{\gamma-1}_{L^{\gamma-1}_T W^{k,\widehat{\gamma}}}\|v\|_{L^\infty_T H^k} \lesssim \\ &\leq \|v_0\|_{H^k} + M(T)^\gamma . \end{aligned} \tag{3.1}$$

Since $\gamma > 1$, M is a continuous function and $M(0) = \|v_0\|_{H^k}$, taking $\|v_0\|_{H^k}$ small enough allows us to find a global bound on $M(T)$ and to ensure global existence for the solution v of (NLS).

We would like to follow a similar path for the general problem. However, we have to work more, because the forcing term in (NLSP) depends on x and t and, above all, for fixed t it only has a first-order zero in $w = 0$. If we try to use the same argument as above, we fail to establish

$$\|f(w+\tilde{u}) - f(\tilde{u})\|_{L^1_T H^k} \lesssim C(\|\tilde{u}\|)\|w\|^\alpha$$

for some $\alpha > 1$ and some choice of the two norms of $\tilde{u}$ and w, because the term on the left side contains a term of the type $f'(\tilde{u})w$.
Luckily, $f'(\tilde{u})$ is both fast-decaying in time and linear on w, and we will use that to our advantage.

We can rewrite the forcing term using Taylor expansion

$$g[w] = f(w+\tilde{u}) - f(\tilde{u}) = V[\tilde{u}] \cdot w + F[w,\tilde{u}] \cdot w^2,$$

where we wrote in short the first order term of the Taylor series and the second order remainer:

$$\begin{aligned} V[\tilde{u}] \cdot w &= \frac{\partial f}{\partial z}(\tilde{u})w + \frac{\partial f}{\partial \bar{z}}(\tilde{u})\overline{w}, \\ F[w,\tilde{u}] \cdot w^2 &= w^2 \int_0^1 (1-t)\frac{\partial^2 f}{\partial z^2}(\tilde{u}+tw)\,dt + \\ &\quad + 2w\overline{w} \int_0^1 (1-t)\frac{\partial^2 f}{\partial z \partial \bar{z}}(\tilde{u}+tw)\,dt + \\ &\quad + \overline{w}^2 \int_0^1 (1-t)\frac{\partial^2 f}{\partial \bar{z}^2}(\tilde{u}+tw)\,dt. \end{aligned} \tag{3.2}$$

Doing a straightforward calculation, we can find the bounds

$$\left|\frac{\partial^l}{\partial z^l}\frac{\partial^m}{\partial \bar{z}^m} f(z)\right| \lesssim_{l+m} |z|^{\gamma-l-m} \quad \forall m,n \in \mathbb{N},\ \gamma \in \mathbb{R}, \tag{3.3}$$

and all the derivatives such that $l+m>\gamma$ are 0 if γ is an odd integer. Because of this property of f, the coefficients of $V[\tilde{u}]$ as an $\mathbb{R}$-linear matrix and, respectively, of $F[w,\tilde{u}]$ as a complex-valued quadratic $\mathbb{R}$-form will satisfy certain boundedness conditions, such as

$$|V[\tilde{u}]| \lesssim |\tilde{u}|^{\gamma-1}, \qquad |F[w,\tilde{u}]| \lesssim \max\{|w|,|\tilde{u}|\}^{\gamma-2}.$$

Now, we can morally treat $V[\tilde{u}] =: V(t,\mathbf{x})$ as a time-dependent potential since it is a linear operator (even if only $\mathbb{R}$-linear) and rewrite equation (NLSP) as

$$i\partial_t w + \Delta w - V(t,x)\cdot w = F[w,\tilde{u}]\cdot w^2. \tag{3.4}$$

The first idea to reach the final goal is to see the above equation as a nonlinear version of the equation

$$\left(i\partial_t + \Delta - V(t)\right) w = F, \tag{3.5}$$

where, in our case, $F = F[t,\mathbf{x},w] = F[w,\tilde{u}]\cdot w^2$. We will refer to this equation as the *linearized equation* when $F=0$. On the left side, we now have a power-two function of w, which could help us to conclude. If we only could establish Strichartz estimates for equation (3.5), we would be able to conclude rewriting all the steps in (3.1) and getting a term $M^2(T)$ in the right side of the inequality.

3.2 Strichartz Estimates for the linearized equation

In this section we study equation (3.5), recalling a useful theorem from [3]. The essence of the original theorem is preserved, though we have to make slight modifications to apply it in this work.
From this point, C_0 denotes the smallest Strichartz constant that is common to all the estimates in Theorem A.1.2 (it is finite for $n \geq 3$ by interpolation,

thanks to the endpoint estimate), while $C(p_1, p_2)$ is the common constant for the estimates with the time exponent ranging from p_1 to p_2.

Theorem 3.2.1 (Nonlinear Schrödinger with potential, [3, Thm. 1.1])**.** *Let $n \geq 2$, let I be an interval, $0 \in I \subseteq [0, +\infty)$, and assume $V(t,x)$ is a $\mathbb{C}$-valued potential belonging to*

$$V(t,x) \in L_I^{r_1} L^{r_2}, \quad \frac{1}{r_1} + \frac{n}{2r_2} = 1,$$

for some fixed $r_1 \in [1, \infty)$ and $r_2 \in (n/2, \infty]$. Let $u_0 \in L^2$ and $F \in L_I^{\tilde{p}'} L^{\tilde{q}'}$ for some admissible pair $(\tilde{p}, \tilde{q})$.
Then the integral equation corresponding to (3.5) has a unique solution $u \in C_I L^2$ which belongs to $L_I^p L^q$ for all admissible pairs (p,q) and satisfies the Strichartz estimates

$$\|u\|_{L_I^p L^q} \leq C_V \|u_0\|_{L^2} + C_V \|F\|_{L_I^{\tilde{p}'} L^{\tilde{q}'}}. \tag{3.6}$$

When $n \geq 3$, the constant C_V can be estimated by $k\,(1 + 2C_0)^k$, where C_0 is the Strichartz constant for the free equation, while k is an integer such that the interval I can be partitioned in k subintervals J with the property $\|V\|_{L_J^{r_1} L^{r_2}} \leq (2C_0)^{-1}$. A similar statement holds when $n = 2$, provided we replace C_0 by $C(p, \tilde{p})$. Finally, if V is $\mathbb{R}$-valued and $F \equiv 0$, the solution satisfies the conservation of energy

$$\|u(t)\|_{L^2} \equiv \|u_0\|_{L^2}, \quad t \in I.$$

The following proof is only for the case $n \geq 3$ (we have the endpoint estimates), $r_1 = 1$ and $r_2 = \infty$, which is the case we are interested in. The conservation of energy in that particular case is not useful in this work, thus the proof is omitted. A complete proof can be found in the article. As it will be clear in the proof, nothing changes if V is of the form we are considering in our problem, i.e.

$$V(t,x) \cdot w(t,x) = v_1(t,x) w(t,x) + v_2(t,x) \overline{w}(t,x), \tag{3.7}$$

$$v_1, v_2 \in L_I^{r_1} L^{r_2} \ \mathbb{C}\text{-valued}. \tag{3.8}$$

Proof. Consider a small interval $J = [0, \delta] \subset I$, and let Z be the Banach space

$$Z = C_J L^2 \cap L^2_J L^{\frac{2n}{n-2}}, \quad \|v\|_Z := \max\left\{\|v\|_{L^\infty_J L^2}, \quad \|v\|_{L^2_J L^{\frac{2n}{n-2}}}\right\}.$$

We will call $p^* = \frac{np}{n-p}$. By interpolation, Z is embedded in all Schrödinger-admissible spaces $L^p_J L^q$. For any $v(t,x) \in Z$ we define the map

$$\Phi(v) = e^{it\Delta}u_0 + \int_0^t e^{i(t-s)\Delta}[F(s) - V(s)v(s)]ds\,.$$

A direct application of the usual Strichartz estimates for Schrödinger equation gives

$$\begin{aligned}\|\Phi(v)\|_{L^p_J L^q} &\leq C_0\,\|u_0\|_{L^2} + C_0\|Vv\|_{L^1_J L^2} + C_0\|F\|_{L^{\tilde{p}'}_J L^{\tilde{q}'}} \leq \\ &\leq C_0\,\|u_0\|_{L^2} + C_0\|V\|_{L^1_J L^\infty}\|v\|_{L^\infty_J L^2} + C_0\|F\|_{L^{\tilde{p}'}_J L^{\tilde{q}'}}\end{aligned} \tag{3.9}$$

for all admissible pairs $(p,q), (\tilde{p},\tilde{q})$. In particular, since Z is continuously embedded in all Strichartz spaces, choosing $(p,q) = (\infty, 2)$ or $(2, 2^*)$ we obtain

$$\|\Phi(v)\|_Z \leq C_0\,\|u_0\|_{L^2} + C_0\|V\|_{L^1_J L^\infty}\|v\|_Z + C_0\|F\|_{L^{\tilde{p}'}_J L^{\tilde{q}'}}$$

Thus, $\Phi(v) \in Z$, since thanks to the Strichartz estimates it is also continuous with values in L^2 (by density).

So, we have $\Phi : Z \to Z$. Now, let's take δ small such that

$$C_0\|V\|_{L^1_J L^\infty} \leq \frac{1}{2}. \tag{3.10}$$

Two consequences: first of all, the mapping Φ is a contraction on Z and hence has a unique fixed point $v(t,x)$ which is the required solution; second, v satisfies

$$\|v\|_{L^p_J L^q} \leq 2C_0\,\|u_0\|_{L^2} + 2C_0\|F\|_{L^{\tilde{p}'}_J L^{\tilde{q}'}}.$$

Now, we can subdivide the interval I in k subintervals, such that in each of these (3.10) holds. Iterating the above argument k times, and noticing that for each $J = [t_0, t_1]$ there's the bound

$$\|v\,(t_1)\|_{L^2} \leq 2C_0\,\|v\,(t_0)\|_{L^2} + 2C_0\|F\|_{L^{\tilde{p}'}_J L^{\tilde{q}'}},$$

we obtain inductively the Strichartz estimates (3.6) and the claimed bound for the Strichartz constant. □

The proof provides an intuition of what happens if we suppose more spacial regularity on u_0 and V. Everything we need besides that is just a product estimate to split the two terms V and v in (3.9).

Corollary 3.2.2. *In the hypotheses of Theorem 3.2.1, suppose $k \in \mathbb{N}$, $u_0 \in H^s$, $V \in L^1_I W^{k,\infty}$, $F \in L^{\tilde{p}'}_I H^{s,\tilde{q}'}$, where $0 \le s \le k$. Then, given two admissible pairs (p,q), $(\tilde{p},\tilde{q})$, the solution given by Theorem 3.2.1 satisfies*

$$\|u\|_{L^p_I H^{s,q}} \le C_{V,s} \|u_0\|_{H^s} + C_{V,s} \|F\|_{L^{\tilde{p}'}_I H^{s,\tilde{q}'}}. \tag{3.11}$$

Proof. The proof is identical to the proof of Theorem 3.2.1, except that we use everywhere the Schrödinger H^k Strichartz estimates and the estimate

$$\|Vv\|_{H^s} \lesssim_{n,s} \|V\|_{W^{k,\infty}} \|v\|_{H^s} \tag{3.12}$$

in (3.9). The above estimate is obtained by complex interpolation between the cases $s = 0$ (Hölder's inequality) and $s = k$ (a generalization of Hölder's inequality that can be easily proved by hand). The bound on $C_{V,s}$ is analogue to the one of C_V with C_0 replaced by $C_0 C_{n,s}$, where $C_{n,s}$ is the implicit constant in the above inequality. □

Remark 3.2.3. Again, nothing changes if we suppose V to be as in (3.7) with suitable regularity assumptions. In fact, we can obtain (3.12) using complex interpolation, since V is a sum of a $\mathbb{C}$-linear operator and an $\mathbb{C}$-antilinear operator for which that estimate holds.

Since the proofs for the other cases where $r_1 > 1$ are in close analogy to the above one, one could also establish the Strichartz estimates (3.11) assuming $V \in L^{r_1}_I H^{s,r_2}$, using for example Proposition A.2.1 insdead of (3.12). Note that in this case V can have non-integer regularity.

3.3 Global theorem

Proof of Theorem 3.0.1. The first part of the theorem has already been proved in Chapter 2. For the second part, we will use the tools we have developed so far.

Step 1. $V[\tilde{u}] \in L^1_{\mathbb{R}} W^{m,\infty}$. This can be verified thanks to the decay of $\tilde{u}$. The estimates for the first-order derivatives of V are obtained in this way (remember that $\tilde{u}$, and so V, are constant in the last $n-2$ variables):

$$\|D_{\mathbf{x}} V\|_{L^1 L^\infty} \lesssim \||\tilde{u}|^{\gamma-2} D_{\mathbf{x}} \tilde{u}\|_{L^1 L^\infty} = \||u|^{\gamma-2} D_x u\|_{L^1 L^\infty(\mathbb{R}^2)} \lesssim$$
$$\lesssim \||u|^{\gamma-2}\|_{L^1 L^\infty(\mathbb{R}^2)} \|D_x u\|_{L^\infty L^\infty(\mathbb{R}^2)} \lesssim \||u|^{\gamma-2}\|_{L^1 L^\infty(\mathbb{R}^2)} \|D_x u\|_{L^\infty H^{1+\varepsilon}(\mathbb{R}^2)} < \infty,$$

where the last inequality is obtained using Proposition 1.2.3 and Corollary 1.2.5. We can similarly compute the other derivatives. Namely, thanks to Corollary A.2.3 and Remark A.2.4), and (3.3), we informally have

$$D^k (V[\tilde{u}]) \sim |\tilde{u}|^{\gamma-2} D^k \tilde{u},$$

and so, by Corollary 1.2.5, we obtain a uniform bound for the derivatives up to order k with the following hypotheses:

$$\|V\|_{L^1_{\mathbb{R}} W^{k,\infty}} < \infty,$$
$$\text{if} \qquad (\gamma > \max\{k, 3\} \vee \gamma \in 2\mathbb{N}_0 + 5) \tag{3.13}$$
$$\text{and} \qquad u_0 \in H^{2,2}(\mathbb{R}^2) \cap H^{k+1+\varepsilon}(\mathbb{R}^2)\,, \tag{3.14}$$

where $\varepsilon > 0$. Hypothesis (3.13) can be weakened to accept the case $\gamma = 3$ provided $u_0 \in H^{2,2}(\mathbb{R}^2) \cap H^{k+2+\varepsilon}(\mathbb{R}^2)$ (this hypothesis seems to be not removable, unless one proves a decay result also for the derivatives of u). Taking k=m, we conclude.

Step 2. Nonlinear estimate.

Proposition 3.3.1. *Let $m \in \mathbb{N}_{\geq 2}$ and $3 \leq \gamma < \chi(m)$ such that either $m+2 < \gamma$ or $\gamma \in 2\mathbb{N}_0 + 3$. If $u_0 \in H^s(\mathbb{R}^2)$, $m+1 < s \in \mathbb{R}$, then we have*

$$\|F[\tilde{u}, w] \cdot w^2\|_{L^1_T H^m} \lesssim C(u) \|w\|^2_{S^m(T)} \left(1 + \|w\|_{S^m(T)}\right)^{\gamma-2},$$

where $C(u)$ depends on some $L^p L^\infty$ norms of u and its derivatives, and

$$\|w\|_{S^m(T)} := \|w\|_{L^\infty_T H^m} + \|w\|_{L^2_T W^{m,2^*}}.$$

If $d = 3$ and $3 \leq \gamma < 5$, we must assume the additional hypothesis

$$s > \lfloor 4m/3 \rfloor + 2. \tag{3.15}$$

Proof. We shall omit the T in all the norms in the proof. The explicit formula for the forcing term is given in (3.2). We are just going to estimate the first of the three terms, since the other two are identical.
First, we want to estimate the quantity

$$\|F[\widetilde{u}, w] \cdot w^2\|_{L^1L^2}.$$

Using (3.3), we have

$$\left| w^2 \int_0^1 (1-t) \frac{\partial^2 f}{\partial z^2}(\widetilde{u} + tw)\, dt \right| \lesssim |w|^2 \int_0^1 (1-t)|\widetilde{u} + tw|^{\gamma-2}\, dt \lesssim$$
$$\lesssim |w|^2 \int_0^1 (1-t)|\widetilde{u}|^{\gamma-2} + |tw|^{\gamma-2}\, dt \lesssim |w|^2|\widetilde{u}|^{\gamma-2} + |w|^{\gamma}.$$

So, we immediately obtain

$$\|F[\widetilde{u}, w] \cdot w^2\|_{L^1L^2} \lesssim \|w\|_{L^\infty L^2} \left(\|w\|_{L^{\gamma-1}L^\infty} \|\widetilde{u}\|^{\gamma-2}_{L^{\gamma-1}L^\infty} + \|w\|^{\gamma-1}_{L^{\gamma-1}L^\infty} \right) \lesssim$$
$$\lesssim \|w\|_{L^\infty H^m} \|w\|_{L_T^{\gamma-1} W^{m,\widehat{\gamma}}} \|\widetilde{u}\|^{\gamma-2}_{L^{\gamma-1}L^\infty} + \|w\|_{L^\infty H^m} \|w\|^{\gamma-1}_{L_T^{\gamma-1} W^{m,\widehat{\gamma}}}.$$

We have used the Sobolev immersions

$$\|w\|_{L^\infty L^2} \leq \|w\|_{L^\infty H^m} \lesssim \|w\|_{S^m}, \tag{3.16}$$
$$\|w\|_{L^{\gamma-1}L^\infty} \leq \|w\|_{L^{\gamma-1} W^{m,\widehat{\gamma}}} \lesssim \|w\|_{S^m}. \tag{3.17}$$

Next, for $|\beta| = m$, we have due to Gagliardo–Nirenberg inequality that

$$D^\beta (F[\widetilde{u}, w] \cdot w^2) \sim w D^\beta w F[\widetilde{u}, w] + D^\beta F[\widetilde{u}, w] \cdot w^2.$$

The first term can be estimated as before with the same bound (up to multiplicative constants). For the second term there is some work to do.

$$\left| D^\beta \int_0^1 (1-t) \frac{\partial^2 f}{\partial z^2}(\widetilde{u} + tw)\, dt \right| \lesssim \int_0^1 (1-t) \left| D^\beta \frac{\partial^2 f}{\partial z^2}(\widetilde{u} + tw) \right| dt.$$

As we did in Proposition 2.1.1, we can estimate

$$\left| D^\beta [D^2 f(\widetilde{u} + tw)] \right|$$

with a sum of terms

$$\left| D^{m+2} f(\widetilde{u} + tw)] \right| \left| \prod_{j=1}^k D^{\beta_j}(\widetilde{u} + tw) \right|, \qquad \text{with} \quad 1 \leq k \leq m, \quad \sum \beta_j = \beta. \tag{3.18}$$

As it will be clear in the proof, the parameter t doesn't play any role in the estimate, so we will omit it from now. The above product can be written explicitly as a big sum. As we already did, we use Remark A.2.4 to suppose that the only terms appearing from the above term are of the form

$$\begin{aligned}
&1)\ w^2(|w|^{\gamma-(k+2)} + |\widetilde{u}|^{\gamma-(k+2)})w^r \widetilde{u}^s D^a w\, D^b \widetilde{u},\\
&2)\ w^2(|w|^{\gamma-(k+2)} + |\widetilde{u}|^{\gamma-(k+2)})w^{k-1} D^m w,\\
&3)\ w^2(|w|^{\gamma-(k+2)} + |\widetilde{u}|^{\gamma-(k+2)})\widetilde{u}^{k-1} D^m \widetilde{u},
\end{aligned}$$

where $0 \le r, s$, $r + s = k - 2$, $a + b = m$. We have used again (3.3).

Let's now subdivide the proof into three cases.

Case 1. $m \ge 2$, $\gamma \ne 3$.
If $d \ge 4$, the first condition is not restrictive. We miss the case $m = 1$, $d = 3$, $\gamma \in [3, 5)$. We can estimate all the terms in the following way:

$$\begin{aligned}
\|(1)\|_{L^1L^2} \le \|w\|_{L^{\gamma-2}L^\infty} \left(\|w\|^{\gamma-(k+2)}_{L^{\gamma-2}L^\infty} + \|\widetilde{u}\|^{\gamma-(k+2)}_{L^{\gamma-2}L^\infty}\right)\cdot\\
\cdot \|w\|^r_{L^{\gamma-2}L^\infty}\|\widetilde{u}\|^s_{L^{\gamma-2}L^\infty}\|D^m w\|_{L^\infty L^2}\|D^m\widetilde{u}\|_{L^\infty L^\infty},
\end{aligned}$$

$$\|(2)\|_{L^1L^2} \le \|w\|^2_{L^{\gamma-1}L^\infty} \left(\|w\|^{\gamma-(k+2)}_{L^{\gamma-1}L^\infty} + \|\widetilde{u}\|^{\gamma-(k+2)}_{L^{\gamma-1}L^\infty}\right) \|w\|^{k-1}_{L^{\gamma-1}L^\infty}\|D^m w\|_{L^\infty L^2},$$

$$\begin{aligned}
\|(3)\|_{L^1L^2} \le \|w\|_{L^\infty L^2}\|w\|_{L^{\gamma-2}L^\infty}\cdot\\
\cdot \left(\|w\|^{\gamma-(k+2)}_{L^{\gamma-2}L^\infty} + \|\widetilde{u}\|^{\gamma-(k+2)}_{L^{\gamma-2}L^\infty}\right) \|\widetilde{u}\|^{k-1}_{L^{\gamma-2}L^\infty}\|D^m\widetilde{u}\|_{L^\infty L^\infty}.
\end{aligned}$$

Since $m \ge 2$, $\gamma > 4$, i.e. $\gamma - 2 > 2$. This means that all the norms of $\widetilde{u}$ in the previous estimates are finite thanks to the decay (1.2.3). Moreover, since $m > \chi^{-1}(\gamma) = \frac{d}{\gamma}$, the following estimate hold together with (3.16) and (3.17)

$$\|w\|_{L^{\gamma-2}L^\infty} \le \|w\|_{L^{\gamma-2}W^{m,\widehat{\gamma-1}}} \lesssim \|w\|_{S^m}.$$

From all the above estimates, the stated estimate is proved.

Case 2. $\gamma = 3$.
In this case, only survive the terms where $k = 1$, i.e. the terms we have to estimate are the following ones:

$$(1') \quad w^2 D^m w, \qquad\qquad (2') \quad w^2 D^m \widetilde{u}.$$

For the first term:

$$\|(1')\|_{L^1L^2} \lesssim \|w\|^2_{L^2L^\infty}\|D^m w\|_{L^\infty L^2} \overset{(3.17)}{\lesssim} \|w\|^3_{S^m(T)}.$$

For the second term, suppose first $d \geq 4$.

$$\|(2')\|_{L^1L^2} \lesssim \|w\|^2_{L^2L^4}\|D^m\widetilde{u}\|_{L^\infty L^\infty} \lesssim \|w\|^2_{S^m(T)}\|D^m\widetilde{u}\|_{L^\infty L^\infty},$$

where in the last inequality we used Sobolev immersions and the fact that $2^* \leq 4 < \infty$.
If $d = 3$, then we must necessarily assume that

$$D^m\widetilde{u} \in L^4L^\infty\,,$$

which is true thanks to hypothesis (3.15) and Gagliardo–Nirenberg. Then we have (note that $(8/3, 4)$ is an admissible pair if $d = 3$):

$$\|(2')\|_{L^1L^2} \lesssim \|w\|^2_{L^{8/3}L^4}\|D^m\widetilde{u}\|_{L^4L^\infty} \lesssim \|w\|^2_{S^m(T)}\|D^m\widetilde{u}\|_{L^4L^\infty}.$$

Case 3. $m = 1$, $d = 3$, $3 < \gamma < 5$.
Of course, we don't have the term (1), since $k = 1$. We can estimate (2) as we did in case 1. Then, there are two terms left to estimate:

$$(1'')\quad w^2|\widetilde{u}|^{\gamma-3}D^m\widetilde{u}, \qquad\qquad (2'')\quad w^{\gamma-1}D^m\widetilde{u}.$$

We have for $\gamma > 3 + 1/4$:

$$\|(1'')\|_{L^1L^2} \lesssim \|w^2\|^2_{L^{8/3}L^4}\|\widetilde{u}\|^{\gamma-3}_{L^{4(\gamma-3)}L^\infty}\|D\widetilde{u}\|_{L^\infty L^\infty}\,,$$

while for $\gamma \geq 3 + 1/3$ (i.e., when $\widehat{\gamma} \leq 2(\gamma - 1)$):

$$\|(2'')\|_{L^1L^2} \lesssim \|w\|^{\gamma-1}_{L^{\gamma-1}L^{2(\gamma-1)}}\|D\widetilde{u}\|_{L^\infty L^\infty} \lesssim \|w\|^{\gamma-1}_{L^{\gamma-1}W^{1,\widehat{\gamma}}}\|D\widetilde{u}\|_{L^\infty L^\infty}.$$

In the remaining cases, we must assume some decay of $D\widetilde{u}$ in time. It will be enough to have another derivative bounded, so we can assume $s > 3$ (which coincides with (3.15)). With this condition, thanks to Gagliardo–Nirenberg, we have

$$\|D\widetilde{u}(t)\|_{L^\infty} \lesssim \langle t\rangle^{-\frac{1}{2}}\,.$$

So, we can proceed as follows for $(1'')$:

$$\|(1'')\|_{L^1L^2} \lesssim \|w^2\|^2_{L^{8/3}L^4}\|\widetilde{u}\|^{\gamma-3}_{L^\infty L^\infty}\|D\widetilde{u}\|_{L^4L^\infty}.$$

For $(2'')$, let's call

$$p = \frac{4}{3}\frac{\gamma-1}{\gamma-2},$$

chosen in order to have the Strichartz couple $(p, 2(\gamma-1))$ (note that $2(\gamma-1) < 2^* = 6$). We will call

$$q = p/(\gamma-1) = \frac{4}{3(\gamma-2)} \in (1, 4/3).$$

Finally, note that $q' > 4$. We can now estimate as below:

$$\begin{aligned}\|(2'')\|_{L^1L^2} &\lesssim \|w^{\gamma-1}\|_{L^qL^2}\|D\widetilde{u}\|_{L^{q'}L^\infty} \lesssim \\ &\lesssim \|w\|^{\gamma-1}_{L^pL^{2(\gamma-1)}}\|D\widetilde{u}\|_{L^{q'}L^\infty} \lesssim \|w\|^{\gamma-1}_{S^m(T)}\|D\widetilde{u}\|_{L^{q'}L^\infty},\end{aligned}$$

and conclude the proof also for the last case. □

Step 3. Conclusion. We know that a local solution always exists in a time interval that depends only on $\|w_0\|_{H^m}$, hence it is sufficient to find an *a priori* estimate for the H^m norm of the solution.
As we did at the beginning of the chapter, we suppose to have a local solution w in $[0,T]$ in the hypotheses of the second part of Theorem 3.0.1. Then,

$$\begin{aligned}\|w\|_{S^m(T)} &\lesssim \|w_0\|_{H^m} + \|F[\widetilde{u}, w]\cdot w^2\|_{L^1_TH^m} \lesssim \\ &\lesssim \|w_0\|_{H^m} + \|w\|^2_{S^m(T)}\left(1+\|w\|_{S^m(T)}\right)^{\gamma-2},\end{aligned} \tag{3.19}$$

where the first inequality follows from Corollary 3.2.2, and the second one is given by Proposition 3.3.1.
Since $u(t)$ is a continuous function with values in H^m and all the Strichartz norms of the solutions must be locally finite in time, $\|w\|_{S^m(T)}$ is a continuous function of T and

$$\lim_{T\to 0^+}\|w\|_{S^m(T)} = \|w_0\|_{H^m}.$$

Assuming $\|w_0\|_{H^m} < \varepsilon$, where ε is taken small enough, $\|w\|_{S^m(T)}$ is forced to be uniformly bounded in time. Namely, called $C \geq 1$ the implicit constant in the inequality (3.19), taking $\varepsilon = \frac{1}{2^{\gamma+1}C}$ we have that the inequality

$$C(\varepsilon + x^2(1+x)^{\gamma-2}) - x \geq 0$$

is false in a whole interval (x_1, x_2), $\varepsilon < x_1 < x_2$. Thus, necessarily

$$\|w\|_{S^m(T)} \leq x_1 \quad \forall T > 0,$$

and this completes the proof. $\square$

There is another way to conclude without using the theory of NLS with time-dependent potentials. The proof is more elementary, since it doesn't use the results in Section 3.2, even though the technique is almost the same.

Step 3 reloaded. This time we work on the original equation (NLSP). To find an *a priori* estimate for the H^m norm of the solution, we do a similar calculation, but using the usual Strichartz estimates. Moreover, we choose a time interval $I = [t_1, T]$, $t_1 \geq 0$, and we call $J = [t_1, +\infty)$. We have:

$$\begin{aligned} \|w\|_{S^m(I)} &\lesssim \|w_0\|_{H^m} + \|V[\tilde{u}] \cdot w\|_{L^1_I H^m} + \|F[\tilde{u}, w] \cdot w^2\|_{L^1_I H^m} \lesssim \\ &\lesssim \|w_0\|_{H^m} + \|V[\tilde{u}]\|_{L^1_J W^{k,\infty}} \|w\|_{S^m(I)} + \\ &+ \|w\|^2_{S^m(I)} \left(1 + \|w\|_{S^m(I)}\right)^{\gamma-2}. \end{aligned}$$

The second one is again given by Proposition 3.3.1.
We notice that $\|V[\tilde{u}]\|_{L^1_J W^{k,\infty}}$ converges to 0 as $t_1 \to +\infty$ thanks to step 1. Again, $\|w\|_{S^m(I)}$ is a continuous function of T and

$$\lim_{T \to t_1^+} \|w\|_{S^m(I)} = \|w(t_1)\|_{H^m}.$$

We now assume $t_1 \gg 0$ such that $\|V[\tilde{u}]\|_{L^1_J W^{k,\infty}} < \varepsilon$, and $\|w(t_1)\|_{H^m} < \varepsilon$, where ε is taken small enough. Then, the solution is again uniformly bounded in time on the interval J, since the inequality to be satisfied becomes

$$C(\varepsilon + \varepsilon x + x^2(1+x)^{\gamma-2}) - x \geq 0,$$

which is false in a certain interval (x_1, x_2), $\varepsilon < x_1 < x_2$.
The previous argument gives us a global solution provided $\|w(t_1)\|_{H^m} < \varepsilon$. But this is true if we start with an initial datum such that $\|w_0\|_{H^m} \ll \varepsilon$ thanks to Lemma 2.2.4.

As we can see from Proposition 3.3.1, the argument for the global well-posedness works for every couple (m, γ) where $m \geq 1$ is an integer and $3 \leq \gamma < \chi(m)$. Thus, Theorem 3.0.1 holds also in these cases, provided we also prove local well-posedness. This will briefly be discussed in Chapter 4.

Chapter 4

General case and possible developements

This chapter has a different style from the others. Here we try to collect some possible developements and generalizations of this problem, and try to make an *a priori* analysis to understand in which cases this problem could be interesting or worth studying with tools similar to those we have used so far. For a complete analysis, new ideas will be required, for instance some regularization method. Nothing can exclude that many improvements on the assumptions can be made by varying the involved techniques, but we think it might be appropriate and meaningful to understand how much we have done in terms of the tools we used.

4.1 Local well-posedness in the subcritical case

As we said at the end of Chapter 3, local well-posedness in the general case $m \in \mathbb{N}_{\geq 1}$ and $3 \leq \gamma < \chi(m)$ still has to be done explicitly. Despite that, it should remain true provided the nonlinearity is regular enough. To see this, we have to look at the terms one has to estimate while proving the bound for the non linear term of equation (NLSP), namely (2.11) and (2.12). The main point here (we will look at (2.12) only for the sake of simplicity), after simplifying, is to control the following quantity on the time interval $[0,T]$:

$$\| |\zeta^{(0)}|^{\gamma-k} \prod_{j=1}^{k} D^{\beta_j} \zeta^{(j)} \|_{L^{\widetilde{p}'} L^{\widetilde{q}'}}, \tag{4.1}$$

where $(\widetilde{p},\widetilde{q})$ is a Strichartz pair, the $\zeta^{(j)}$ are ϕ, ψ, or $\widetilde{u}$, and exactly one of them is $\phi-\psi$, except for $\zeta^{(0)}$. The argument is the following.

(1) If the term $\widetilde{u}$ does not appear, then we can control the above quantity in the following way:

$$\lesssim T^{\alpha} \| |\zeta^{(0)}|^{\gamma-k} \prod_{j=1}^{k} D^{\beta_j} \zeta^{(j)} \|_{L^{r} L^{\widetilde{q}'}} \lesssim \|\zeta^{(0)}\|^{\gamma-k}_{L^{p_0} L^{s_0}} \prod_{j=1}^{k} \|D^{\beta_j} \zeta^{(j)}\|_{L^{p_j} L^{s_j}} ,$$

where there exist $r \geq \widetilde{q}$ (in order to apply Hölder's inequality) and $\{(p_j,q_j)\}_{j=0}^{k}$ Strichartz pairs such that

$$\frac{1}{s_j} = \frac{1}{q_j} - \frac{m-|\beta_j|}{d},$$

setting $|\beta_0|=0$. This is well known to be true whenever $\gamma \leq \chi(m)$ for a suitable couple $(\widetilde{p},\widetilde{q})$ as a part of the classical NLS theory (see for example [7, Lemma 5.1]), since that is the usual way to obtain local well-posedness for the classical problem. With that, we can continue and use Sobolev immersions to obtain

$$\begin{aligned} &\lesssim T^{\alpha} \|\zeta^{(0)}\|^{\gamma-k}_{L^{p_0} W^{m,q_0}} \prod_{j=1}^{k} \|\zeta^{(j)}\|_{L^{p_j} W^{m,q_j}} \lesssim \\ &\qquad \lesssim T^{\alpha} \|\phi-\psi\|_{S^m(T)} \left(\|\phi\|_{S^m(T)} + \|\psi\|_{S^m(T)} \right)^{\gamma-1}. \end{aligned}$$

(2) For the general case, we estimate first all the terms with $\widetilde{u}$ and its derivatives using the $L^\infty L^\infty$ norm. Then, what remains is a term of the form (4.1), but where γ has been reduced by a certain quantity. So, instead of γ, we have a certain power $1 \leq l < \gamma$, which obviously satisfies $l < \chi(m)$, i.e. we can establish the same estimate as above with a smaller power. Actually, m has been reduced too, but this is not a problem since the argument in the previous step also works if $1 \leq \sum |\beta_j| \leq m$.

Always remaining in the hypothesis of integer regularity m, one can consider the critical case $\gamma = \chi(m)$. This should be feasible to study, also considering that there aren't any problems with the regularity: in fact, assuming $\gamma \geq 3$, the only critical cases are

$$\begin{cases} \gamma = 3,\ m = \dfrac{d}{2} - 1 & \text{when } d \text{ is an even number,} \\ \gamma = 5,\ m = \dfrac{d-1}{2} & \text{when } d \text{ is an odd number.} \end{cases}$$

Unfortunately, the proof we used in Chapter 3 relies heavily on the hypothesis $\gamma < \chi(m)$, so it would require a proof on its own.

Finally, as a completion of this work, it would be interesting to study the subcritical and critical cases in the fractional regularity regime $s \in \mathbb{R}^+$, using nonlinear estimates for fractional derivatives (see [11, §5.4.3], and [7, appendix]). This will become imperative in the generalization to higher dimensions described in the next section, since, as we will see, γ is bounded from above.

4.2 Generalizations

Higher dimensions

Here we consider a natural generalization of the problem discussed in this work. Instead of making u_0 depend on two variables, we can fix $n \in \mathbb{N}_{\geq 1}$, $n < d$, and consider the function u_0 on the domain $\mathbb{R}^n$, and thence its extension $\widetilde{u}_0$ exactly as we did in the case $n = 2$. One could ask if the problem still has a global solution, and what are the conditions for this to happen. First, let's have a look at the n-dimensional solution u. What we needed from the beginning was a global solution u whose $W^{k,\infty}$ norm is bounded and integrable. The natural generalization of the required conditions is

$$\chi_n(0) \leq \gamma < \chi_n(1). \tag{4.2}$$

The second inequality allows us to find a global solution in $H^1(\mathbb{R}^n)$ using conservation of mass and energy. The first inequality, as we have already noticed in Section 1.2, is necessary to apply the argument with the pseudo-conformal transform and obtain the decay for u: namely, we obtain that

$$\|u(t)\|_{L^\infty} \lesssim \langle t \rangle^{-n/2}$$

if $u_0 \in H^{k,k}(\mathbb{R}^n)$, $k > n/2$, in order to control the L^∞ norm. There are not additional conditions on γ to apply Corollary 1.2.5, since $2/n < \chi_n(0) - 1 \leq \gamma - 1$ for every $n \in \mathbb{N}_{\geq 1}$, and the fact that $\gamma - 1$ could become less than 1 is not a problem, as we can see from the proof of Proposition 1.1.8.

Then, there are some conditions involving the dimension d. We cite directly [7] for some reasonable and quite general conditions for this problem to be locally well-posed in some space $H^s(\mathbb{R}^d)$, $s > 0$, assuming that g depends only on $u(t,x)$, and not on x and t (in what follows, $F(\zeta)$ is our $g(z)$).

($F1$) *(smoothness)* $F \in C^{\{s\}}(\mathbb{C};\mathbb{C})$, *with* $F(0) = 0$.

($F2$) *(growth rate for large* ζ*)*

- *If* $s > m/2$, *no assumption.*
- *If* $s \leq m/2$ *and if* $F(\zeta)$ *is a polynomial in* ζ *and* $\bar{\zeta}$, *then* $\text{degree}(F) = k \leq \chi(s)$.
- *If* $s \leq m/2$ *and if* F *is not a polynomial, then* $F(\zeta) = O^{\{s\}}\left(|\zeta|^k\right)$ *as* $|\zeta| \to \infty$ *where* k *is a finite number such that*
$$\{s\} \leq k \leq \chi(s).$$

Here, $\{s\} = \lceil s \rceil$ for $s > 0$, and $F(\zeta) = O^{\{s\}}\left(|\zeta|^k\right)$ means that

$$D^i F(\zeta) = O\left(|\zeta|^{k-i}\right), \quad i = 0, 1 \ldots \{s\}. \tag{4.3}$$

In our case, this means that γ must satisfy

$$\begin{cases} \gamma \leq \chi(s) & \text{if } \gamma \in 2\mathbb{N}_0 + 3, \\ \lceil s \rceil < \gamma \leq \chi(s) & \text{otherwise.} \end{cases}$$

Clearly, the conditions on γ are satisfied fore some $s > 0$ if and only if they hold for $s = \chi^{-1}(\gamma)$. This means that we can write our final conditions as

$$\lceil \chi_d^{-1}(\gamma) \rceil < \gamma \qquad \text{if } \gamma \notin 2\mathbb{N}_0 + 3, \tag{4.4}$$

while there are no conditions if $\gamma \ni 2\mathbb{N}_0 + 3$.

The case $\gamma \in 2\mathbb{N}_0 + 3$ is simple: $\gamma \neq 3$ for $n = 1$, no conditions on γ for $n = 2$ (as we saw), and $\gamma = 3$ for $n = 3$. Concerning the other case, it can be shown by hand (and with some numerical help) that (4.4) corresponds to

$$\begin{cases} \gamma > 1 & d \leq 6, \\ \gamma \in (1, \chi_d(1)] \cup (2, 7/3] \cup (\lfloor d/2 \rfloor, +\infty) & d = 7, \\ \gamma \in (1, \chi_d(1)] \cup (\lfloor d/2 \rfloor, +\infty) & d = 8, 9, \\ \gamma \in (1, \chi_d(1)] \cup (\lceil d/2 \rceil, +\infty) & d \geq 10. \end{cases}$$

Considering together conditions (4.2) and (4.4), we can see when their intersection is non-empty. We can visualize this in Table 4.1, where every coloured square strictly above the diagonal corresponds to a non-empty intersection. Each colour here means a way the two conditions interact. For example,

Admissible couples

	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1														
2														
3														
4														
5														
6														
7														
8														
9														
10														
11														

Table 4.1: Table of admissible couples

the yellow zone is where condition (4.4) vanishes, and γ only has to satisfy (4.2). The pattern continues as one can imagine with these two infinitely long stripes, namely $n \in \{1, 2\}$ and $d \in \{n+1, n+2\}$. The second row is the case we have considered in the previous chapter.

Actually, we have to pay attention to the fact that condition (4.4) is likely too weak: in Chapter 2, we had to assume $\gamma \geq m + 1$ because we needed the m^{th} derivative to be Lipschitz continuous, because g is a subtraction of f evaluated at two different functions. So, a more reasonable condition would be

$$\lceil \chi_d^{-1}(\gamma) \rceil + 1 < \gamma. \tag{4.5}$$

Luckily, this is completely equivalent to

$$\lceil \chi_{d+2}^{-1}(\gamma) \rceil < \gamma,$$

and this has the effect to shift the whole table by two columns on the left, eliminating the second stripe $d \in \{n+1, n+2\}$ completely. In the end, the only admissible cases are $n = 1, 2$, where condition (4.2) is weak because the energy critical-power is $+\infty$, and $d \leq 4$, where condition (4.5) vanishes, plus some sporadic cases for $n = 3$ or $d = 7$.

The above were just local considerations. Concerning the globality for small perturbations, a minimal condition (see [7], §6) should be $\chi_d(0) \leq \gamma$, regardless of s. This is automatically true in the conditions considered above, since $\chi_d(0) < \chi_n(0)$.

Critical observations. The analysis we have just made in this subsection is vague, and nothing can exclude that all these conditions are necessary. First, the first inequality in (4.4) is not always true: for instance, there can be local well posedness for classical NLS equation in H^2 with just C^1 regularity (see [7], §4 for some references). Secondly, the first inequality in (4.2) is a condition we wanted for the decay of the solution u to (NLS). The cubic equation in one dimension doesn't satisfy (4.2) and is an example of problem with such a decay only for small data (see [6]), so it's not interesting in the same way, but it is worth mentioning.

We talked about the hypothesis $\gamma \leq \chi_n(1)$ in the introduction. For a long time, the question whether a solution to the supercritical defocusing (NLS) for large data is always global in time or not remained unsolved. Global well-posedness in this case was conjecture among others by Bourgain[1]. After some partial results, the first counterexample to this conjecture was given in [10] in a small number of cases, namely $(n, \gamma) \in \{(5, 9), (6, 5), (8, 3), (9, 3)\}$. This is an evidence that nothing guarantees a global solution in the energy-supercritical case, and this is why the assumption we made seems to be sharp[2], or at least reasonable considering the current state of the art in the field.

[1] Bourgain, J., *Problems in Hamiltonian PDE'S*, Geom. Funct. Anal. (2000), Special Volume, Part I, 32–56. First Problem of Section 3.

[2] Actually, what should be checked is the energy-critical case $\gamma = \chi_n(1)$, which is actually much more complex. A good reference for this topic is [8], in the same book of [12].

Appendix A

A.1 The linear Schrödinger equation

The original Strichartz estimates in the form below were obtained by various authors, among which Ginibre and Velo, in the non-endpoint case, while the endpoint case was obtained by Keel and Tao in 1996.

Definition A.1.1. Let $p, q \in [2, +\infty]$. The pair (p, q) is called *(Schrödinger-) admissible* in dimension d if $(p, q, d) \neq (2, \infty, 2)$ and

$$\frac{2}{p} + \frac{d}{q} = \frac{d}{2}.$$

If $p = 2$, the pair (p, q) is called the *endpoint* case.

Theorem A.1.2 (Strichartz estimates)**.** *Let (p, q), $(\widetilde{p}, \widetilde{q})$ be two admissible couples in dimension d. Then the following estimates hold for $f \in L^2(\mathbb{R}^d)$ and $F \in L_t^{\widetilde{p}'} L_{\mathbf{x}}^{\widetilde{q}'}$:*

$$\|e^{it\Delta} f\|_{L_t^p L_{\mathbf{x}}^q} \lesssim_{d,p,q} \|f\|_{L_{\mathbf{x}}^2},$$
$$\|\int_s^t e^{-is\Delta} F(s, \cdot)\, ds\|_{L_{\mathbf{x}}^2} \lesssim_{d,p,q} \|F\|_{L_t^{\widetilde{p}'} L_{\mathbf{x}}^{\widetilde{q}'}},$$
$$\|\int_0^t e^{i(t-s)\Delta} F(s, \cdot)\, ds\|_{L_t^p L_{\mathbf{x}}^q} \lesssim_{d,p,q,p',q'} \|F\|_{L_t^{\widetilde{p}'} L_{\mathbf{x}}^{\widetilde{q}'}}.$$

For $d \neq 2$, the multiplicative constants can be chosen to depend only on d.

Remark A.1.3. Since $e^{it\Delta}$ commutes with derivations, the above estimates also hold replacing everywhere $L_{\mathbf{x}}^r$ with $H^{s,r}$, $s \in \mathbb{R}$. For instance,

$$\|e^{it\Delta} f\|_{L_t^p H_{\mathbf{x}}^{s,q}} \lesssim_{d,p,q} \|f\|_{H_{\mathbf{x}}^s}.$$

A.2 Product and Chain rules

Let's define the Fourier multipliers $D^s = (-\Delta)^{s/2},\ J^s = (1-\Delta)^{s/2}$.

Proposition A.2.1 (Kato–Ponce estimates, [4, §1])**.** *Let $u, v \in \mathscr{S}\left(\mathbb{R}^d\right)$. Then,*

$$\|D^s(uv)\|_{L^r\left(\mathbb{R}^d\right)} \lesssim \|D^s u\|_{L^{p_1}} \|v\|_{L^{p_2}} + \|u\|_{L^{q_1}} \|D^s v\|_{L^{q_2}}\ ,$$
$$\|J^s(uv)\|_{L^r\left(\mathbb{R}^d\right)} \lesssim \|J^s u\|_{L^{p_1}} \|v\|_{L^{p_2}} + \|u\|_{L^{q_1}} \|J^s v\|_{L^{q_2}}\ ,$$

whenever the following conditions hold:

$$\frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q_1} + \frac{1}{q_2}, \quad p_j, q_j \in (1, \infty], \quad s > \max\left(0, \frac{d}{r} - d\right) \text{ or } s \in 2\mathbb{N}^+\,.$$

Proposition A.2.2 (Gagliardo–Nirenberg inequality, [9, §12.5])**.** *Let $1 \le p, q \le \infty, m \in \mathbb{N}^+, k \in \mathbb{N}_0$, with $0 \le k < m$, and let θ, r be such that*

$$0 \le \theta \le 1 - k/m$$

and

$$(1-\theta)\left(\frac{1}{p} - \frac{m-k}{d}\right) + \theta\left(\frac{1}{q} + \frac{k}{d}\right) = \frac{1}{r} \in [0,1].$$

Then there exists a constant $c = c(m, d, p, q, \theta, k) > 0$ such that

$$\|\nabla^k u\|_r \le c\|u\|^{\theta}_{L^q\left(\mathbb{R}^d\right)} \|\nabla^m u\|^{1-\theta}_{L^p\left(\mathbb{R}^d\right)} \tag{A.1}$$

for every $u \in L^q\left(\mathbb{R}^d\right) \cap \dot{W}^{m,p}\left(\mathbb{R}^d\right)$, with the following exceptional cases:

1. *If $k = 0, mp < d$, and $q = \infty$, we assume that u vanishes at infinity.*
2. *If $1 < p < \infty$ and $m - k - d/p$ is a non-negative integer, then (A.1) only holds for $0 < \theta \le 1 - k/m$.*

Corollary A.2.3. *Let $l_j \ge 0$, $j = 1, \dots, k$ integers, and $m = \sum_{j=1}^k l_j$. Assume $p, q, r \in [1, +\infty]$ such that*

$$\frac{1}{r} = \frac{1}{p} + \frac{k-1}{q}\,.$$

Then, for every $\varphi \in L^q\left(\mathbb{R}^d\right) \cap \dot{W}^{m,p}\left(\mathbb{R}^d\right)$, we have that

$$\|\prod_{j=1}^k D^{l_j}\varphi\|_{L^r(\mathbb{R}^d)} \lesssim_{d,m,p,q,r} \|\varphi\|^{k-1}_{L^q(\mathbb{R}^d)} \|D^m\varphi\|_{L^p(\mathbb{R}^d)}.$$

Proof. Assume $k \geq 2$, as the case $k = 1$ is trivial. Suppose initially $r < \infty$. Let $\theta_j \in \mathbb{R}$ such that

$$\frac{\theta_j}{r} = \frac{l_j}{m}\left(\frac{1}{p} - \frac{m - l_j}{d}\right) + \left(1 - \frac{l_j}{m}\right)\left(\frac{1}{q} + \frac{l_j}{d}\right) = \frac{l_j}{m}\frac{1}{p} + \left(1 - \frac{l_j}{m}\right)\frac{1}{q}.$$

Hence, we can use Gagliardo–Nirenberg estimate to obtain

$$\|D^{l_j}\varphi\|_{L^{r/\theta_j}(\mathbb{R}^d)} \lesssim \|\varphi\|_{L^q(\mathbb{R}^d)}^{1-l_j/m}\|D^m\varphi\|_{L^p(\mathbb{R}^d)}^{l_j/m}.$$

Note that we are not in the two exceptional cases of Theorem A.2.2: if $l_j = 0$, the estimate is trivial. Clearly, $\theta_j \in [0, 1]$, and $\sum_{j=1}^k \theta_j = 1$. Using this fact and the previous inequality, we get to

$$\|\prod_{j=1}^k D^{l_j}\varphi\|_{L^r(\mathbb{R}^d)} \leq \prod_{j=1}^k \|D^{l_j}\varphi\|_{L^{r/\theta_j}(\mathbb{R}^d)} \lesssim \prod_{j=1}^k \|\varphi\|_{L^q(\mathbb{R}^d)}^{1-l_j/m}\|D^m\varphi\|_{L^p(\mathbb{R}^d)}^{l_j/m},$$

and the inequality is proved.
If $r = \infty$, then $p, q = \infty$ and the proof follows in a similar way. □

Remark A.2.4. Corollary A.2.3 tells us that, if we have a sum of terms of the form $\prod_{j=1}^k D^{l_j}\varphi$ in a certain L^r norm, one can assume without loss of generality that all the terms appearing in the sum satisfy $l_j = m\delta_{j,1}$. In other words, there are some dominant terms in the sum, and they correspond to those where all the derivatives fall on a single function. As an example, one has that $D^m[u^\gamma] \sim u^{\gamma-1}D^m u$ for $\gamma > k$, in the sense that we can control the L^r norm of the left side in the same way we control the norm of the right side.

One can use Gagliardo–Nirenberg inequality in the same way to obtain the Kato–Ponce estimates, at least for integer s. This is part of a general principle that we state below, which helps quite well understanding the technique involved despite its vagueness.

Principle A.2.5 (Top order terms dominate; [14, §A])**.** *When distributing derivatives, the dominant terms are usually the terms in which all the derivatives fall on a single factor; if the factors have unequal degrees of smoothness, the dominant term will be the one in which all the derivatives fall on the roughest (or highest frequency) factor.*

A.3 Notations

- $\lceil a \rceil$ is the smallest integer greater or equal to a.
- $\lfloor a \rfloor$ is the greatest integer smaller or equal to a.
- $u_t = \partial_t u = \frac{\partial u}{\partial t}$ is the derivative w.r.t. the variable t, whether classical or distributional.
- $p^* = \frac{dp}{d-p}$ when positive or infinite, whenever working in $\mathbb{R}^d$.
- usually, $s \in [0, \infty)$ and $k, m \in \mathbb{N}_0$ when these are regularity exponents of some Sobolev spaces.
- $L^p_I X$, where $p \in [1, +\infty]$ and X is a Banach space, is the space of L^p functions on the interval I with values in X. We write $L^p_T X$ if the interval I is either $[0, T]$, $T > 0$, or $[T, 0]$, $T < 0$.
- $A \lesssim B$ means that $A \leq CB$, where C is a constant which depends only on certain parameters, which can appear as subscripts of the symbol.
- $\langle x \rangle := (1 + |x|^2)^{1/2}$, $x \in \mathbb{R}^d$.
- $W^{k,p}(\mathbb{R}^d)$, $k \in \mathbb{N}_0$ and $1 \leq p \leq +\infty$, are the usual *Sobolev spaces*.
- $H^{s,p}(\mathbb{R}^d)$, $s \in \mathbb{R}$ and $1 < p < +\infty$, are the *Bessel potential spaces* $\langle D \rangle^{-s} L^p(\mathbb{R}^d)$.

Bibliography

[1] T. Cazenave, *Semilinear Schrödinger equations*, Courant Institute of Mathemetical Sciences, 2003.

[2] T. Cazenave, F. B. Weissler, *The Cauchy problem for the critical nonlinear Schrödinger equation in H^s*, Nonlinear Analysis **14** (1990), 807–836.

[3] P. D'Ancona, V. Pierfelice, N. Visciglia, *Some remarks on the Schrödinger equation with a potential in $L^r_t L^s_x$*, Math. Ann. **333** (2005), no. 2, 271–290.

[4] P. D'Ancona, *A short proof of commutator estimates*, J. Fourier Anal. Appl. **25** (2019), no. 3, 1134–1146.

[5] J. Ginibre, G. Velo, *On a class of nonlinear Schrödinger equations. I. The Cauchy problem, general case*, J. Funct. Anal. **32** (1979), no. 1, 1–32.

[6] M. Ifrim, D. Tataru, *Global bounds for the cubic nonlinear Schrödinger equation (NLS) in one space dimension*, Nonlinearity **28** (2015), 2661–2675.

[7] T. Kato, *On Nonlinear Schrödinger Equations, II. H^s-Solutions and Unconditional Well-Posedness*, J. Anal. Math. **67** (1995), 281–306.

[8] R. Killip, M. Vişan, *Nonlinear Schrödinger equations at critical regularity*, Lecture notes — G. Staffilani et al., Clay Mathematics Proceedings V.17, *Evolution Equations*, AMS for CMI, 2013.

[9] G. Leoni, *A First Course in Sobolev Spaces*, Second edition, AMS, 2017.

[10] F. Merle, P. Raphaël, I. Rodnianski, and J. Szeftel, *On blow up for the energy super critical defocusing nonlinear Schrödinger equations*, preprint, arXiv:1912.11005.

[11] T. Runst, W. Sickel, *Sobolev Spaces of Fractional Order, Nemytskij Operators, and Nonlinear Partial Differential Equations*, Walter de Gruyter, 1996.

[12] G. Staffilani, *The theory of nonlinear Schrödinger equations*, Lecture notes — G. Staffilani et al., Clay Mathematics Proceedings V.17, *Evolution Equations*, AMS for CMI, 2013.

[13] T. Tao, *Finite time blowup for a supercritical defocusing nonlinear Schrödinger system*, Anal. PDE **11** (2018), no. 2, 383–438.

[14] T. Tao, *Nonlinear Dispersive Equations, Local and Global Analysis*, AMS, 2006.

[15] M. E. Taylor, *Partial differential equation III. Nonlinear equations*, 2nd ed, Springer, 2001.