Equivariant motivic Hall algebras

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Motivation

Let $L|K$ be a finite, totally ramified extension of complete discretely valued fields of characteristic $(0,p)$ with perfect residue field k. A smooth projective variety X over L comes with the following linear algebraic data.

- The crystalline cohomology groups of the special fibre X_k become K-vector spaces after inverting p.
- These carry a natural Frobenius action, compatibly with a fixed lift σ of the Frobenius of k to K.
- By comparison with the de Rham cohomology of X, they inherit a (Hodge) filtration over L.

Definition 1: Let isoc_K denote the category of F-isocrystals over K, that is, the category of pairs (V,φ), where $V\in\mathrm{vect}_K$ is a finite dimensional vector space over K, and $\varphi\colon V\otimes_{K,\sigma}K\xrightarrow{\sim}V$ a σ-semilinear automorphism of V. The category of ***filtered*** F***-isocrystals*** over $L|K$ is defined to be the fibre product

$$\mathrm{Fil}^{\mathbb{Z}}_{L|K}\times_{\mathrm{vect}_K}\mathrm{isoc}_K=\mathrm{Fil}^{\mathbb{Z}}_L\times_{\mathrm{vect}_L}\mathrm{isoc}_K, \tag{1}$$

where $\mathrm{Fil}^{\mathbb{Z}}_{L|K}$ is the category of finite dimensional K-vector spaces together with a $\mathbb{Z}$-filtration over L.

In fact, the p-adic analogue of a ***period domain*** over $\mathbb{C}$, parametrizing Hodge structures, is a moduli space for ***semistable*** filtered F-isocrystals over K, cf. Definition 2. They form an abelian subcategory of (1), which Colmez and Fontaine describe in terms of certain representations of the absolute Galois group,

$$\mathrm{rep}^{\mathrm{cris}}_K(G_L)\xrightarrow{\sim}(\mathrm{Fil}^{\mathbb{Z}}_L\times_{\mathrm{vect}_L}\mathrm{isoc}_K)^{\mathrm{ss}}_0.$$

The cohomology of p-adic period domains is studied in [2]. A similar strategy is pursued in various settings.

- Originally, by Harder and Narasimhan, in the context of moduli spaces of vector bundles on curves.
- Reineke [5] counts $\mathbb{F}_q$-points of quiver moduli spaces in order to infer their Betti numbers (over $\mathbb{C}$).
- Joyce [4] refines this point count to motivic measures of more general moduli spaces, over any field K.

Our goal is to generalize this approach to accommodate for the equivariant setting of [2]. Consider three quasi-abelian K-linear categories $\mathcal{E}$, $\mathcal{B}$, and $\mathcal{D}$, and assume that $\mathcal{B}$ and $\mathcal{D}$ are semisimple. Let

$$\mathcal{E}\xrightarrow{\omega}\mathcal{B}\xleftarrow{\nu}\mathcal{D}$$

be two K-linear ***exact isofibrations***. Then for a field extension $L|K$, we replace (1) by the fibre product

$$(\mathcal{E}_L\times_{\mathcal{B}_L}\mathcal{B})\times_{\mathcal{B}}\mathcal{D}=\mathcal{E}_L\times_{\mathcal{B}_L}\mathcal{D},\ \text{ where }\mathcal{E}_L:=\mathcal{E}\otimes_K L. \tag{2}$$

Example 1: (a) For a quiver Q, the fibre functor $\mathrm{rep}_{\mathcal{B}}(Q)\to\mathcal{B}$, $M\mapsto\bigoplus_{i\in Q_0}M_i$, is an exact isofibration.

(b) Similarly, this applies to the forgetful functor on the category of representations in $\mathcal{B}$ of a group G. By arguing pointwise, this extends to (pro-)group schemes over K. In fact, it follows that

$$\mathrm{Fil}^{\mathbb{Z}}_K\xrightarrow{\omega}\mathrm{vect}_K\xleftarrow{\nu}\mathrm{isoc}_K,$$

and indeed, the fibre functor of a ***quasi-Tannakian category*** over K is an exact K-linear isofibration.

(c) In the same vein, this works for the functor $\omega\colon\mathrm{Fil}^{\Lambda}_{\mathcal{B}}\to\mathcal{B}$, where Λ is a totally ordered abelian group.

Slope filtrations and Hall algebras

It is explained in [1] how to express the aforementioned notions of semistability in our categorical setting.

Definition 2: Let $\mathcal{E}^{\simeq}$ denote the maximal subgroupoid. Suppose $\mathcal{E}$ is equipped with two maps as follows.

- The ***rank function*** $\mathrm{rk}\colon\pi_0(\mathcal{E}^{\simeq})\to\mathbb{N}$, additive on exact sequences, such that $\mathrm{rk}(E)=0\Leftrightarrow E=0$.
- A ***degree function*** $\deg\colon K_0(\mathcal{E})\to\Lambda$, with $\deg(E)\le\deg(E')$ for all $E\to E'$ with (co-)kernel $=0$.

Then E is ***semistable*** if $\mu(N)\le\mu(E)$ for all $0\ne N\le E$, where $\mu(E)=\frac{\deg E}{\mathrm{rk}\,E}\in\Lambda_{\mathbb{Q}}$ is the ***slope*** of E.

The full subcategory $\mathcal{E}^{\mathrm{ss}}_{\lambda}$ of semistable objects of $\mathcal{E}$ of slope $\in\{\lambda,\infty\}$ is inherently abelian.

Example 2: (a) For the category $\mathcal{E}$ of vector bundles on a (connected) smooth projective curve X over the field K, we have the usual notions of rank and degree. Note that $\mathcal{O}_X\twoheadrightarrow\mathcal{O}_X(1)$ has (co-)kernel $=0$.

(b) Let Q be a (connected) quiver, and $\mathcal{E}=\mathrm{rep}_K(Q)$. Then $\mathrm{rk}(M):=\sum_{i\in Q_0}\dim_K(M_i)$ for $M\in\mathcal{E}$, and any choice of $\theta\in\Lambda^{Q_0}$ defines a degree function via $\deg_\theta\colon K_0(\mathcal{E})\xrightarrow{\dim}\mathbb{Z}^{\oplus Q_0}\xrightarrow{\theta}\Lambda$.

(c) Let $\mathcal{E}=\mathrm{Fil}^{\Lambda}_{L|K}$ with rank function $\mathrm{rk}(V,F^\bullet)=\dim_K(V)$. The degree is weighted by the jumps of $F^\bullet$,

$$\deg_\bullet(V,F^\bullet)=\sum_{\lambda\in\Lambda}\lambda\cdot\dim_L(F^\lambda V/F^{\lambda-1}V)\in\Lambda.$$

On isoc_K, define $\deg_\sigma(V,\varphi):=-\mathrm{val}_p(\det\varphi)$. The fibre product in (1) is endowed with $\deg:=\deg_\bullet+\deg_\sigma$.

Several further examples appear in the survey article [1], where the following is proved in this generality.

Proposition 1: *There is a unique filtration* $F^\bullet\colon\Lambda^{\mathrm{op}}_{\mathbb{Q}}\times\mathcal{E}\to\mathcal{E}$, *such that* $0\subsetneq F^{\lambda_1}E\subsetneq\ldots\subsetneq F^{\lambda_n}E=E$, *for* $E\in\mathcal{E}$, *is uniquely determined by* $F^{\lambda_i}E/F^{\lambda_{i-1}}E$ *being semistable of decreasing slopes* $\lambda_1>\ldots>\lambda_n$.

If $K=\mathbb{F}_q$ and $\mathcal{E}$ is finitary, we can express Proposition 1 as an equation in the ***Hall algebra*** of $\mathcal{E}$. This is the convolution algebra $\mathrm{H}(\mathcal{E})=\mathbb{Q}[\pi_0(\mathcal{E}^{\simeq})]$ of finitely supported $\mathbb{Q}$-valued functions on $\pi_0(\mathcal{E}^{\simeq})$, that is,

$$(f*g)(E)=\sum_{N\le E}f(N)g(E/N),\ \text{ for }f,g\in\mathbb{Q}[\pi_0(\mathcal{E}^{\simeq})].$$

More precisely, if we complete $\mathrm{H}(\mathcal{E})$ with respect to its $K_0(\mathcal{E})$-grading, where $\mathbb{1}_E$ lies in degree $[E]$, then

$$\mathbb{1}_{\pi_0(\mathcal{E}^{\simeq})}=\sum_{\lambda_1>\ldots>\lambda_n}\mathbb{1}_{\pi_0(\mathcal{E}^{\mathrm{ss}\simeq}_{\lambda_1})}*\ldots*\mathbb{1}_{\pi_0(\mathcal{E}^{\mathrm{ss}\simeq}_{\lambda_n})}\in\widehat{\mathrm{H}}(\mathcal{E}). \tag{3}$$

If $\mathcal{E}$ is ***hereditary***, the Euler form $\chi(M,N)=\dim\mathrm{Hom}(M,N)-\dim\mathrm{Ext}^1(M,N)$ defines a twisted group ring $\mathbb{Q}^{\langle\chi\rangle}[K_0(\mathcal{E})]$, with $[M][N]=q^{-\chi(N,M)}[N\oplus M]$. By [5], Lemma 6.1, there is an ***integration morphism***

$$\int_{\mathcal{E}}\colon\widehat{\mathrm{H}}(\mathcal{E})\longrightarrow\mathbb{Q}^{\langle\chi\rangle}[[K_0(\mathcal{E})]],\ \mathbb{1}_E\longmapsto\tfrac{1}{\#\,\mathrm{Aut}(E)}[E]. \tag{4}$$

Integrating (3) yields a formula (5), counting points of the ***moduli stack of objects*** $\mathcal{M}_\alpha$ of class $\alpha\in K_0(\mathcal{E})$.

Equivariant motivic Hall algebras

Over an arbitrary field K, the idea is to replace the number of points $q=\#\mathbb{A}^1(\mathbb{F}_q)$ by the affine line itself. To this end, we understand it as an element $\mathbb{L}=[\mathbb{A}^1_K]\in K_0(\mathrm{Var}/K)$ of the Grothendieck ring of varieties.

Definition 3: Let $\mathcal{Z}$ be a stack in groupoids on the big fppf-site $\mathrm{Aff}^{\mathrm{fppf}}_K$ of affine schemes over K. Then the (relative) ***Grothendieck ring of stacks*** $K_0(\mathrm{Sta}/\mathcal{Z})$ is the free $\mathbb{Z}$-module on geometric equivalence classes of algebraic stacks over $\mathcal{Z}$, of finite type and with affine stabilizers over K, modulo the relations

$$[\mathfrak{X}\amalg\mathfrak{X}']=[\mathfrak{X}]+[\mathfrak{X}'],$$
$$[\mathfrak{X}_1]=[\mathfrak{X}_2]\text{ for all locally trivial Zariski fibrations }\mathfrak{X}_i\to\mathfrak{X}_0\text{ with equivalent fibres.}$$

This ensures that $K_0(\mathrm{Var}/K)[\mathbb{L}^{-1},(\mathbb{L}^n-1)^{-1}\mid n\in\mathbb{N}]\xrightarrow{\sim}K_0(\mathrm{Sta}/K)$. As above, there is a HN-recursion

$$[\mathcal{M}_\alpha]=\sum_{\substack{\alpha_1>\ldots>\alpha_n\\ \alpha_1+\ldots+\alpha_n=\alpha}}\mathbb{L}^{-\sum_{i<j}\chi(\alpha_j,\alpha_i)}[\mathcal{M}^{\mathrm{ss}}_{\alpha_1}]\cdots[\mathcal{M}^{\mathrm{ss}}_{\alpha_n}]\in K_0(\mathrm{Sta}/K). \tag{5}$$

This makes sense, assuming $\dim_K\mathrm{Hom}_{\mathcal{E}}(-,-)<\infty$, by the following finiteness result.

Theorem 1: *Let* $-\,\widehat{\otimes}_K\,-$ *denote the* K*-linear Cauchy completion of the tensor product. The functor*

$$\mathcal{M}\colon\mathrm{Aff}^{\mathrm{fppf}}_K\longrightarrow\mathrm{Grpd},\ \mathrm{Spec}(A)\longmapsto(\mathcal{E}\,\widehat{\otimes}_K\,A)^{\simeq},$$

defines an algebraic stack, locally of finite type over K, *called the moduli stack of objects of* $\mathcal{E}$.

The ***motivic Hall algebra*** $\mathcal{H}(\mathcal{E}):=K_0(\mathrm{Sta}/\mathcal{M})$ of $\mathcal{E}$ is the convolution algebra along the correspondence

$$\mathcal{M}\times_K\mathcal{M}\xleftarrow{(\partial_2,\partial_0)}\mathcal{S}_2(\mathcal{E})\xrightarrow{\partial_1}\mathcal{M}, \tag{6}$$

where $\mathcal{S}_2(\mathcal{E})$ is the moduli stack of short exact sequences in $\mathcal{E}$, which are mapped in (6) to their outer terms and their middle term, respectively. That is, multiplication in $\mathcal{H}(\mathcal{E})$ is defined as the composition

$$K_0(\mathrm{Sta}/\mathcal{M})\otimes K_0(\mathrm{Sta}/\mathcal{M})\xrightarrow{-\times_K-}K_0(\mathrm{Sta}/\mathcal{M}\times_K\mathcal{M})\xrightarrow{(\partial_2,\partial_0)^*}K_0(\mathrm{Sta}/\mathcal{S}_2(\mathcal{E}))\xrightarrow{(\partial_1)_*}K_0(\mathrm{Sta}/\mathcal{M}).$$

Let $\mathcal{N}$ be the moduli stack of $\mathcal{B}$. By replacing $K_0(\mathrm{Sta}/-)$ by its $\mathcal{D}$-equivariant variant $K_0^{\mathcal{D}}(\mathrm{Sta}/-)$, we get

$$[\mathcal{M}\times_{\mathcal{N}}\underline{\mathcal{D}}^{\simeq}]\in\widehat{\mathcal{H}}^{\mathcal{D}}(\mathcal{E}),\ \text{ where }\mathcal{H}^{\mathcal{D}}(\mathcal{E})=K_0^{\mathcal{D}}(\mathrm{Sta}/\mathcal{M})\cong\bigoplus_{D\in\pi_0(\mathcal{D}^{\simeq})}K_0^{\mathrm{Aut}(D)}(\mathrm{Sta}/\mathcal{M}),$$

the ***equivariant motivic Hall algebra*** of $\mathcal{E}$, with parabolic induction product between the summands.

Theorem 2: *There is a natural map of simplicial stacks* $\mathcal{S}(\mathcal{E})\to\underline{K_0(S_\bullet(\mathcal{E}))}$, *whose pushforward*

$$\int_{\mathcal{E}}^{\mathcal{D}}\colon\widehat{\mathcal{H}}^{\mathcal{D}}(\mathcal{E})\longrightarrow K_0^{\mathcal{D}}(\mathrm{Sta}/K)^{\langle\chi\rangle}[[K_0(\mathcal{E})]]$$

is an algebra morphism if $\mathcal{E}$ *is hereditary. For* $\mathcal{D}=0$, *this recovers the motivic version of* (4) *in [4].*

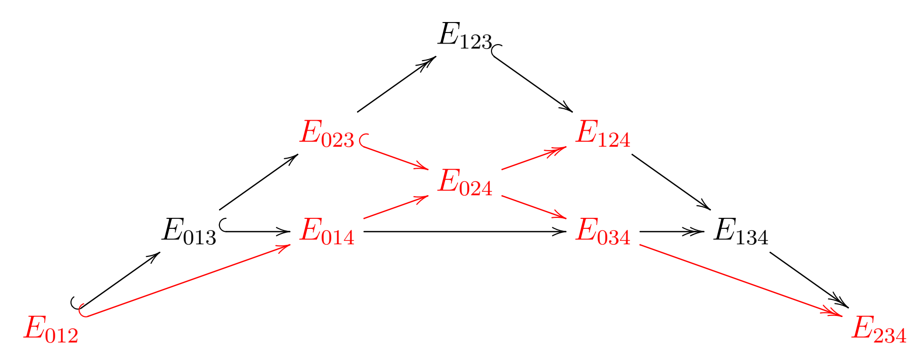
Further directions

If $\mathcal{E}$ carries a duality structure, there is a module over the Hall algebra of $\mathcal{E}$ on isometry classes of selfdual objects, due to M. Young. We have an analogue of Theorem 2 for the ***equivariant motivic Hall module***. In general, we replace $K_0^{\mathcal{D}}(\mathrm{Sta}/-)$ with a ring of analytic stacks (on affinoid spaces) over a *non-Archimedean* field. This again yields a Hall algebra, since Waldhausen's S-construction defines a ***2-Segal stack*** (cf. [3]).

Definition 4: Let $k\ge0$. The n-cells $S_n^{\langle k\rangle}(\mathcal{E})$ of the ***higher Waldhausen S-construction*** are defined as the full subcategory of the category of diagrams $E\colon\mathrm{Fun}([k],[n])\longrightarrow\mathcal{E},(\beta\colon[k]\to[n])\longmapsto E_\beta$, with

- *(degeneracies)* for every functor $\alpha\colon[k-1]\to[n]$, we have $E_{s^*_{k-1}\alpha}=\ldots=E_{s^*_0\alpha}=0$, and
- *(faces)* for every $\gamma\colon[k+1]\to[n]$, the sequence $E_{d^*_{k+1}\gamma}\hookrightarrow E_{d^*_k\gamma}\longrightarrow\ldots\longrightarrow E_{d^*_1\gamma}\twoheadrightarrow E_{d^*_0\gamma}$ is exact.

Hesselholt and Madsen introduced $S_\bullet^{\langle2\rangle}(\mathcal{E})$ in the context of real algebraic K-theory. We illustrate an element of its 4-skeleton, with image under the upper 3-Segal map $u\colon S_4^{\langle2\rangle}(\mathcal{E})\longrightarrow S_3^{\langle2\rangle}(\mathcal{E})\times_{S_2^{\langle2\rangle}(\mathcal{E})}S_3^{\langle2\rangle}(\mathcal{E})$ in red.



If $\mathcal{E}$ is abelian, u is an equivalence, but this case is an outlier; the general result is as follows.

Theorem 3: *The simplicial category* $S_\bullet^{\langle k\rangle}(\mathcal{E})$ *is a* $2k$*-Segal object.*

References

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