Solutions for exercises, Algebra I (Commutative Algebra) – Week 4

Exercise 15. (Scalar extension of Ext and Tor)
Remember that a module P is projective if and only if it is a direct summand of a free module i.e. $\oplus_{i\in I}A \simeq P \oplus Q$ for a A-module Q and a set I. Then we get $\oplus_{i\in I}B \simeq (\oplus_{i\in I}A)\otimes_A B \simeq P\otimes B \oplus Q\otimes B$; thus $P\otimes B$ is a again a projective module.

1. For Tor_i: Let us begin by a lemma

   Lemma 16. Let M, N be A-modules and $f : A \to B$ a ring homomorphism. Then there is a natural isomorphism of B-modules $M\otimes_A N\otimes_A B \simeq (M\otimes_A B)\otimes_B (N\otimes_A B)$

   Beweis. To prove the Lemma, let us define $\pi : M\otimes_A N\otimes_A B \to M\otimes_A B \times N\otimes_A B$ by $(m\otimes b, n\otimes b') \mapsto m\otimes n\otimes bb'$. It is elementary to check that π is a B-bilinear homomorphism. Now given a B-bilinear homomorphism $f : M\otimes_A B\times(N\otimes_A B)\to P$, define $\overline{f} : M\otimes_A N\otimes_A B \to P$ by $m\otimes n\otimes b \mapsto f(m\otimes 1, n\otimes b)$. Then $\overline{f}$ is B-linear (because f is linear in the second argument) and $\overline{f}\circ\pi(m\otimes b, n\otimes b') = \overline{f}(m\otimes n\otimes bb') = f(m\otimes 1, n\otimes bb') = bf(m\otimes 1, n\otimes b') = f(m\otimes b, n\otimes b')$ using that f is B-bilinear. Thus we have the claimed isomorphism. □

   Let us take a projective resolution of N:

   $$\cdots \xrightarrow{d_{i+1}} P_i \xrightarrow{d_i} \cdots \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} N \to 0$$

   Since B is a flat A-module, the functor $-\circ B$ is exact so the sequence

   $$\cdots \xrightarrow{d_{i+1}\otimes \mathrm{id}_B} P_i\otimes B \xrightarrow{d_i\otimes\mathrm{id}_B} \cdots \xrightarrow{d_1\otimes\mathrm{id}_B} P_0\otimes_A B \xrightarrow{\epsilon\otimes\mathrm{id}_B} N\otimes_A B \to 0$$

   is exact. Thus it yields a projective resolution of $N\otimes_A B$. We can use these resolutions to compute the Tor_i groups, namely the following sequences are exact (resp. in Mod_B and Mod_A; with $d_0 = 0$):

   $$0\to \mathrm{im}(\mathrm{id}_{M\otimes B}\otimes d_{i+1}\otimes\mathrm{id}_B)\to \ker(\mathrm{id}_{M\otimes B}\otimes d_i\otimes \mathrm{id}_B)\to \mathrm{Tor}_i^B(M\otimes B, N\otimes B)\to 0 \quad (*)$$

   $$0\to \mathrm{im}(\mathrm{id}_M\otimes d_{i+1})\to\ker(\mathrm{id}_M\otimes d_i)\to \mathrm{Tor}_i^A(M,N)\to 0 \quad (**)$$

   Since B is flat, tensoring (**) with B gives an exact sequence (in Mod_B):

   $$0\to \mathrm{im}(\mathrm{id}_M\otimes d_{i+1})\otimes_A B\to \ker(\mathrm{id}_M\otimes d_i)\otimes_A B\to \mathrm{Tor}_i^A(M,N)\otimes_A B\to 0$$

   so to get $\mathrm{Tor}_i^A(M,N)\otimes_A B \simeq \mathrm{Tor}_i^B(M\otimes B, N\otimes B)$, it is sufficient to prove that $\ker(\mathrm{id}_M\otimes d_i)\otimes_A B\simeq \ker(\mathrm{id}_{M\otimes B}\otimes d_i\otimes\mathrm{id}_B)$ and $\mathrm{im}(\mathrm{id}_M\otimes d_{i+1})\otimes_A B\simeq \mathrm{im}(\mathrm{id}_{M\otimes B}\otimes d_{i+1}\otimes \mathrm{id}_B)$.

   Let us begin with $\ker(\mathrm{id}_M\otimes d_i)\otimes_A B\simeq\ker(\mathrm{id}_{M\otimes B}\otimes d_i\otimes\mathrm{id}_B)$: We have, by definition the exact sequence:

   $$0\to\ker(\mathrm{id}_M\otimes d_i)\to M\otimes P_i \xrightarrow{\mathrm{id}_M\otimes d_i} M\otimes P_{i-1}$$

Solutions to be handed in before Monday May 4, 4pm.

so tensoring with the flat module B, we get:

$$0 \to \ker(\mathrm{id}_M \otimes d_i) \otimes_A B \to M \otimes P_i \otimes_A B \overset{\mathrm{id}_M \otimes d_i \otimes \mathrm{id}_B}{\to} M \otimes P_{i-1} \otimes_A B$$

The Lemma yields

$$0 \to \ker(\mathrm{id}_M \otimes d_i) \otimes_A B \to (M \otimes B) \otimes_B (P_i \otimes_A B) \overset{\mathrm{id}_{M\otimes B} \otimes d_i \otimes \mathrm{id}_B}{\to} (M \otimes B) \otimes_B (P_{i-1} \otimes_A B)$$

so that we get $\ker(\mathrm{id}_M \otimes d_i) \otimes_A B \simeq \ker(\mathrm{id}_{M\otimes B} \otimes d_i \otimes \mathrm{id}_B)$.

Similarly, the cokernel is defined by the exact sequence:

$$M \otimes P_{i+1} \to M \otimes P_i \to \mathrm{Coker}(d_{i+1}) \to 0$$

so tensoring with B we get:

$$(M \otimes B) \otimes_B (P_{i+1} \otimes B) \to (M \otimes B) \otimes_B (P_i \otimes B) \to \mathrm{Coker}(d_{i+1}) \otimes B \to 0$$

so that $\mathrm{Coker}(d_{i+1}) \otimes B \simeq \mathrm{Coker}(\mathrm{id}_{M\otimes B} \otimes d_{i+1} \otimes \mathrm{id}_B)$. Moreover the image is defined by the exact sequence:

$$0 \to \mathrm{im}(d_{i+1}) \to M \otimes P^i \to \mathrm{Coker}(d_{i+1}) \to 0.$$

By flatness of B, we get the exact sequence

$$0 \to \mathrm{im}(d_{i+1}) \otimes B \to (M \otimes B) \otimes_B (P^i \otimes B) \to \mathrm{Coker}(d_{i+1}) \otimes B \to 0$$

using the previous isomorphism we get $\mathrm{im}(d_{i+1}) \otimes B \simeq \mathrm{im}(\mathrm{id}_{M\otimes B} \otimes d_{i+1} \otimes \mathrm{id}_B)$ So comparing (**)$\otimes B$ with (*), we get $\mathrm{Tor}_i^B(M \otimes B, N \otimes B) \simeq \mathrm{Tor}_i^A(M, N) \otimes_A B$.

2. For Ext^i: As reported the isomorphisms for the Ext^i's do not exist without the assumption M finitely generated. So let us prove the isomorphisms with the assumption M finitely generated (for a counter-example related to the one given on the forum see after the proof).
Moreover, we assume here that: A is noetherian. It will be defined later but the feature that we will use is that for such rings, submodules of finitely generated modules are finitely generated. Let us begin by the Lemma

Lemma 17. Let F be a finitely generated free A-module and N a A-module. Let $f : A \to B$ be ring homomorphism. Then there is a natural isomorphism $\mathrm{Hom}_A(F, N) \otimes_A B \simeq \mathrm{Hom}_B(F \otimes B, N \otimes B)$.

Beweis. Write $F \simeq \oplus_{i=1}^n Ae_i$. Then for any N, we have an isomorphism η_N: $\mathrm{Hom}_A(F, N) \otimes B \simeq (\Pi_{i=1}^n N) \otimes B \simeq (\oplus_{i=1}^n N) \otimes B \simeq \oplus_{i=1}^n (N \otimes B) \simeq \Pi_{i=1}^n (N \otimes B) \mathrm{Hom}_B(F \otimes B, N \otimes B)$. Moreover (check) they give a natural transformation between $\mathrm{Hom}_A(F, -) \otimes B : Mod_A \to Mod_B$ and $\mathrm{Hom}_B(F \otimes B, - \otimes B) : Mod_A \to Mod_B$ i.e. for any homomorphism of A-modules $f : M \to N$, we have $\eta_N \circ (\mathrm{Hom}_A(F, -) \otimes B)(f) = (\mathrm{Hom}_B(F \otimes B, - \otimes B))(f) \circ \eta_M$. □

Following the recipe indicated after Corollary 5.5, one can construction a resolution of M by free (thus projective) A-modules of finite rank: By assumption, one can choose a finite set of generators $m_1, \ldots, m_n$ of M as A-module. We can construct a surjective homomorphism of A-modules $\epsilon : \oplus_{i=1}^n Ae_i = F^0 \to M$ by (extend linearly) $e_i \mapsto m_i$. Since A is noetherian, its kernel is again finitely generated (as submodule of the finitely generated A-module $\oplus_{i=1}^n Ae_i$) so we can find a free A-module of finite rank F^1 that surjects unto $\ker(\epsilon)$: $d^{1'} : F^1 \to \ker(\epsilon)$. Then $F^1 \overset{d^1 = i_{\ker(\epsilon)} \circ d^{1'}}{\to} F^0 \overset{\epsilon}{\to} M \to 0$ is exact and

$\ker(d^1)$ is again finitely generated. So by induction, we get a (projective) resolution of M by free A-modules of finite rank:

$$\cdots \xrightarrow{d_{i+1}} F^i \xrightarrow{d_i} \cdots \xrightarrow{d_1} F^0 \xrightarrow{\epsilon} M \to 0$$

with F^i free for any i.
Then, since B is a flat A-module, we get a resolution of $M \otimes B$ by finite free (thus projective) B-modules:

$$\cdots \xrightarrow{d_{i+1}} F^i \otimes_A B \xrightarrow{d^i \otimes \mathrm{id}_B} \cdots \xrightarrow{d^1 \otimes \mathrm{id}_B} F^0 \otimes B \xrightarrow{\epsilon \otimes \mathrm{id}_B} M \otimes B \to 0$$

So we have

$$0 \to \mathrm{im}(- \circ d^i) \to \ker(- \circ d^{i+1}) \to \mathrm{Ext}^i(M, N) \to 0$$

and

$$0 \to \mathrm{im}(- \circ (d^i \otimes \mathrm{id}_B)) \to \ker(- \circ (d^{i+1} \otimes \mathrm{id}_B)) \to \mathrm{Ext}^i(M \otimes B, N \otimes B) \to 0$$

so to get the isomorphisms, it is sufficient to prove that $\ker(- \circ d^{i+1}) \otimes B \simeq \ker(- \circ (d^{i+1} \otimes \mathrm{id}_B))$ and $\mathrm{im}(- \circ d^i) \otimes B \simeq \mathrm{im}(- \circ (d^i \otimes \mathrm{id}_B))$.

The following exact sequence defines the kernel:

$$0 \to \ker(- \circ d^{i+1}) \to \mathrm{Hom}_A(F^i, N) \xrightarrow{- \circ d^{i+1}} \mathrm{Hom}_A(F^{i+1}, N)$$

so tensoring by the flat A-module B and using the Lemma we get:

$$0 \to \ker(- \circ d^{i+1}) \otimes B \to \mathrm{Hom}_B(F^i \otimes B, N \otimes B) \to \mathrm{Hom}_B(F^{i+1} \otimes B, N \otimes B)$$

i.e. $\ker(- \circ d^{i+1}) \otimes B \simeq \ker(- \circ (d^{i+1} \otimes \mathrm{id}_B))$. And tensoring the exact sequence

$$\mathrm{Hom}_A(F^{i-1}, N) \xrightarrow{- \circ d^i} \mathrm{Hom}_A(F^i, N) \to \mathrm{Coker}(- \circ d^i) \to 0$$

with B, we get $\mathrm{Coker}(- \circ d^i) \otimes B \simeq \mathrm{Coker}(- \circ (d^i \otimes \mathrm{id}_B))$.
Finally tensoring the exact sequence

$$0 \to \mathrm{im}(- \circ d^i) \to \mathrm{Hom}_A(F^i, N) \to \mathrm{Coker}(- \circ d^i) \to 0$$

with the flat A-module B, and using the Lemma, we get $\mathrm{im}(- \circ d^i) \otimes B \simeq \mathrm{im}(- \circ (d^i \otimes \mathrm{id}_B))$ hence the isomorphism.

Counter-example 18. Take $A = \mathbb{Z}$, $B = \mathbb{Q}$ and $M = \mathbb{Q}$ and $N = \mathbb{Z}$. Then B is a flat A-module (follow the proof of Exercise 12 (ii) or it will proved soon that localization gives flat algebras) but $\mathrm{Ext}^1(M, N) \otimes Q \neq 0$ whereas, since B is a field (i.e. any B-module=vector space is free hence projective) $\mathrm{Ext}^1(M \otimes Q, N \otimes Q) = 0$.

Beweis. Let us begin with the following fact (look here if you want to see a proof)

Lemma 19. If an abelian group M is divisible (i.e. for any $m \in M$ and $k \in \mathbb{N}^{>0}$, there is a $m' \in M$ such that $m = km'$) then it is an injective $\mathbb{Z}$-module.

Now we have the natural exact sequence of abelian groups:

$$0 \to \mathbb{Z} \to \mathbb{Q} \xrightarrow{\pi} \mathbb{Q}/\mathbb{Z} \to 0$$

and since $\mathbb{Q}$ is divisible and $\mathbb{Q}/\mathbb{Z}$ is also divisible (as quotient of a divisible group), the sequence is actually an injective resolution of $\mathbb{Z}$. So the Ext^i are given by the cohomology groups of the sequence

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Q}) \xrightarrow{\pi \circ -} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Q}/\mathbb{Z}) \to 0$$

i.e. $\mathrm{Ext}^0(\mathbb{Q},\mathbb{Z}) \simeq \ker(\pi\circ -)$ and $\mathrm{Ext}^1(\mathbb{Q},\mathbb{Z}) \simeq \mathrm{Coker}(\pi\circ -)$. To prove the claim, we just have to exhibit a homomorphism $\mathbb{Q}\to\mathbb{Q}/\mathbb{Z}$ not coming from a homomorphism $\mathbb{Q}\to\mathbb{Q}$.
We have $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q},\mathbb{Q}) \overset{f\mapsto f(1)}{\simeq} \mathbb{Q}$.
Let us define $(\mathbb{Q}/\mathbb{Z})_2 = \{x\in\mathbb{Q}/\mathbb{Z},\ 2^k x = 0 \text{ for some } k>0\}$. It is a subgroup of $\mathbb{Q}/\mathbb{Z}$. It can be proved that it is actually a direct summand of $\mathbb{Q}/\mathbb{Z}$. Let us write $\mathbb{Q}/\mathbb{Z}\simeq(\mathbb{Q}/\mathbb{Z})_2\oplus R$. Consider the homomorphism (check) $\alpha : (\mathbb{Q}/\mathbb{Z})_2 \to (\mathbb{Q}/\mathbb{Z})_2$ defined by $\overline{\frac{a}{2^k}} \mapsto \overline{\frac{a}{2^{k+1}}}$. Then $f\circ\pi \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q},\mathbb{Q}/\mathbb{Z})$, with $f = \alpha\oplus \mathrm{id}_R \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z},\mathbb{Q}/\mathbb{Z})$ cannot be induced by an element of $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q},\mathbb{Q})$. □

Exercise 20. (Properties of elements in polynomial rings)

1. Let $c\in A$ such that $bc=1$ and $n>0$ such that $a^n=0$; then
$$(b+a)c\sum_{i=0}^{n-1}(-ac)^i = (1+ac)\sum_{i=0}^{n-1}(-ac)^i = \sum_{i=0}^{n-1}(-ac)^i - \sum_{i=1}^{n}(-ac)^i = 1-(ac)^n = 1$$
so $b+a$ is invertible.

2. If a_0 is a unit and for any $i>0$, a_i is nilpotent, then a_ix^i is nilpotent in $A[x]$ and since the set of nilpotent elements is an ideal (in particular a group), we get that $\sum_{i=1}^n a_ix^i$ is nilpotent. Then using the first question (in "$A=A[x]$), $f = a_0+\sum_{i=1}^n a_ix^i$ is a unit. To prove the converse, let us prove the following fact:
$$\text{if } f=\sum_{i=0}^n a_ix^i \in A[x] \text{ is a unit, then } a_0 \text{ is a unit and } a_n \text{ (hence } a_nx^n) \text{ is nilpotent} \quad (*)$$
Once (*) established, we conclude by a simple induction: take $f=\sum_{i=0}^n a_ix^n$ a unit, with $n>0$ then by (*), a_0 is a unit in A and a_n is nilpotent. In particular a_nx^n is nilpotent so by the first question $f-a_nx^n$ is again a unit. So if $f-a_nx^n$ is not constant, (*) yields that its leading coefficient, namely a_{n-1}, is nilpotent. So $a_{n-1}x^{n-1}$ is nilpotent; thus by the first question $f-a_nx^n-a_{n-1}x^{n-1}$ is a unit. So by an elementary induction, we get that a_i is nilpotent for $i>0$.
Now to prove (*): take $f=\sum_{i=0}^n a_ix^i$ ($n>0$, $a_n\neq 0$) a unit and $g=\sum_{i=0}^d b_ix^i$ its inverse. We have $1 = fg = \sum_{k=0}^{n+d}(\sum_{i=0}^k a_ib_{k-i})x^k$ and since $A[x]$ is a free A-module, we get $a_0b_0=1$ and for $k>0$, $\sum_{i=0}^k a_ib_{k-i}=0$. The first equality reads a_0 and b_0 are units. Now looking at the term of highest degree x^{n+d}, we have $a_nb_d=0$. Let $k\geq 0$ be an integer such that $a_n^{i+1}b_{d-i}=0$ for any $i\leq k$. Then looking at the coefficient of $x^{n+d-(k+1)}$, since
$$0 = \sum_{i=0}^{n+d-(k+1)} a_ib_{n+d-(k+1)-i} = \sum_{i=n-(k+1)}^{n} a_ib_{n+d-(k+1)-i}$$
since $a_i=0$ for $i>n$ and $b_{n+d-(k+1)-i}=0$ for $n+d-(k+1)-i>d$. So
$$0 = a_nb_{d-(k+1)} + \sum_{i=n-(k+1)}^{n-1} a_ib_{n+d-(k+1)-i}$$
and taking the product by a_n^{k+1}, we get
$$0 = a_n^{k+2}b_{d-(k+1)} + \sum_{i=n-(k+1)}^{n-1} a_ia_n^{k+1}b_{d-((k+1)+i-n)}.$$
But for any $i\leq n-1$, $k+1>((k+1)+i-n)$, so by our induction hypothesis, $a_n^{k+1}b_{d-((k+1)+i-n)}=0$ for any $n-(k+1)\leq i\leq n$. Thus $a_n^{k+2}b_{d-(k+1)}=0$ proving that for any $0\leq k\leq d$, $a_n^{k+1}b_{d-k}=0$. In particular, we get $a_n^{d+1}b_0=0$ hence (product by a_0), $a_n^{d+1}=0$ i.e. the leading coefficient of f is nilpotent as claimed.

3. If all a_i are nilpotent, then all $a_i x^i$ are nilpotent and since $\mathfrak{N}_{A[x]}$ is an ideal we get that $f = \sum_i a_i x^i \in \mathfrak{N}_{A[x]}$.
   To prove the converse, let us prove the following fact:
   $$\text{If } f \in A[x] \text{ is nilpotent, then its coefficient of least degree is nilpotent.} \qquad (**)$$
   Once (**), established, we conclude again by induction, as follows: take $f = \sum_{i=m}^n a_i x^i$ nilpotent, with $a_m \neq 0$ and $a_n \neq 0$. By (**), a_m is nilpotent. Since $\mathfrak{N}_{A[x]}$ is an ideal $f - a_m x^m \in \mathfrak{N}_{A[x]}$. Applying (**) to $f - a_m x^m$ we get that (if it is not 0) a_{m+1} is nilpotent. So by an elementary induction all a_i are nilpotent.
   Now (**) is easy: take $f = \sum_{i=m}^n a_i x^i$ nilpotent, with $a_m \neq 0$ and $a_n \neq 0$ and $\ell > 0$ such $f^\ell = 0$. By a direct calculation, the term of least degree of f^ℓ is $(a_m x^m)^\ell$; thus we get $(a_m x^m)^\ell = 0$ i.e. $a_m^\ell = 0$.

4. If there is a $a \in A$ such that $af = 0 \in A[x]$ then by definition ($A \subset A[x]$), f is a zero-divisor.
   Conversely take $f = \sum_{i=0}^n a_i x^i$ a zero-divisor of degree n (in particular $a_n \neq 0$). Consider the non-zero (since f is a zero-divisor) ideal $\mathfrak{a} \subset A[x]$ given by the annihilator of f i.e. $\mathfrak{a} = \{P \in A[x],\ fP = 0\}$. Take $g = \sum_{i=0}^d b_i x^i \in \mathfrak{a}$ of least degree ($\emptyset \neq \{\deg(P), 0 \neq P \in \mathfrak{a}\} \subset \mathbb{N}$) say $d \neq 0$. If $d = 0$, the claim is true for f.
   So let us analyse the case when $d > 0$. We have $fg = 0$ so looking at the leading term we get that $a_n b_d = 0$; in particular $\deg(a_n g) < d$. But $a_n g f = a_n \cdot 0 = 0$ i.e. $a_n g \in \mathfrak{a}$; by definition of d, $a_n g = 0$. Thus we have $0 = fg = a_n x^n g + \sum_{i=0}^{n-1} a_i x^i g = \sum_{i=0}^{n-1} a_i x^i g$. Looking again at the leading coefficient of $\sum_{i=0}^{n-1} a_i x^i g$, we get $a_{n-1} b_d = 0$; in particular $\deg(a_{n-1} g) < d$ and since $a_{n-1} g f = 0$, by definition of d, we have $a_{n-1} g = 0$. So by an elementary induction we get that $a_i g = 0$ for any i. In particular, looking at the leading term of $a_i g$ we get $b_d a_i = 0$ for any i. Therefore $b_d f = 0$.

Exercise 21. (Short exact sequences)
Let denote $\alpha : M_1 \to M_2$. Let $N_3 \subset M_3$ be a submodule. Then $\ker(\pi) = \pi^{-1}(0) \subset \pi^{-1}(N_3)$. By exactness of the first sequence we have $\mathrm{im}(\alpha) = \ker(\pi) \subset \pi^{-1}(N_3)$ so that we can write $\alpha = i_{N_2/M_2} \circ \overline{\alpha}$ for a homomorphism of A-module $\overline{\alpha} : M_1 \to N_2$ and the inclusion $i_{N_2/M_2} : N_2 \hookrightarrow M_2$. Since π is surjective, the induced homomorphism $\pi_{|N_2} : N_2 \to N_3$ is also surjective: for $n_3 \in N_3 \subset M_3$ take a $m_2 \in M_2$ such that $\pi(m_2) = n_3$ then $m_2 \in \pi^{-1}(n_3) \subset \pi^{-1}(n_3) = N_3$ i.e. $m_2 \in N_2$.
Moreover, $\ker(\pi) = \pi^{-1}(0) \subset \pi^{-1}(N_3) = N_2$ so $\ker(\pi) = \ker(\pi_{|N_2})$. We also have $\mathrm{im}(\overline{\alpha}) = \mathrm{im}(\alpha)$. As a consequence $\mathrm{im}(\overline{\alpha}) = \ker(\pi_{|N_2})$.
The injectivity of $\overline{\alpha}$ follows from $\alpha = i_{N_2/M_2} \circ \overline{\alpha}$ and the injectivity of α.

Exercise 22. (Examples of nilradicals)

1. $A = k[x]$: A is an integral domain, so it does not contain any non-zero nilpotent. So $\mathfrak{N} = (0)$. We have seen in Exercise 16 that an element $f = \sum_i a_i x^i \in k[x]$ is a unit if and only if a_0 is a unit and a_i, for $i > 0$, are nilpotent. Since k is an integral domain (the only nilpotent element is 0), we get that $f \in k[x]$ is a unit if and only if f is a non-zero constant polynomial. Now take $f \in \mathfrak{R}$. We have in particular that $1 + f$ is a unit thus a constant polynomial. So f is a constant polynomial. If $f \neq 0$ then $1 + f(-f^{-1})$ would be invertible but $1 + f(-f^{-1}) = 0$ contradiction. So $f = 0$. i.e. $\mathfrak{R} = (0)$.

2. $A = k[[x]]$: We have seen in Exercise 8 that $Spec(A) = \{(0), (x)\}$ and $MaxSpec(A) = \{(x)\}$. Thus, using Proposition 7.2, we get $\mathfrak{N} = (0) \cap (x) = (0)$ and $\mathfrak{R} = (x)$.

3. $A = k[x]/(x^3)$: we have seen in Exercise 8 that $Spec(A) = \{(x)\} = MaxSpec(A)$ thus $\mathfrak{N} = (x) = \mathfrak{R}$.

4. $A = \mathbb{Z}/18\mathbb{Z}$: We recall that there is a bijection between $Spec(A)$ and $V((18))$. A prime ideal $(p) \subset \mathbb{Z}$ is in $V((18))$ if and only if $p|18$. Therefore $V((18)) = \{(2), (3)\}$ and $Spec(A) = \{(\overline{2}), (\overline{3})\}$. Since $MaxSpec(A) \subset Spec(A)$ and $(\overline{2}) \not\subset (\overline{3})$ nor $(\overline{3}) \not\subset (\overline{2})$, we have $MaxSpec(A) = Spec(A)$. Thus $\mathfrak{N} = (\overline{2}) \cap (\overline{3}) = (\overline{6}) = \mathfrak{R}$.

Exercise 23. (Rings with one prime ideal)

((i)$\Rightarrow$ (ii) Then, by Proposition 7.2 $\mathfrak{N} = \mathfrak{p}$ where $\mathfrak{p}$ is the unique prime ideal of A and since a maximal ideal is prime, $Spec(A) = \{\mathfrak{p}\}$ implies that $\mathfrak{p}$ is maximal. Let $a \in A$. If a is not nilpotent, i.e. $a \notin \mathfrak{p}$, then $\mathfrak{p} \not\subset (a) + \mathfrak{p}$ so $(a) + \mathfrak{p} = A$ by maximality of $\mathfrak{p}$. So there are $b \in A$ and $n \in \mathfrak{p} = \mathfrak{N}$, such that $ab + n = 1$ thus, denoting $\ell > 0$ an integer such that $n^\ell = 0$, we have $a[b(\sum_{i=0}^{\ell-1} n^i)] = (1-n)(\sum_{i=0}^{\ell-1} n^i) = 1 - n^\ell = 1$ i.e. a is a unit.

(ii)$\Rightarrow$(iii) Given a $\overline{a} \in A/\mathfrak{N}\backslash\{0\}$ take a $a \in A$ lifting $\overline{a}$. Then $a \notin \mathfrak{N}$ i.e. a is not nilpotent so, by hypothesis, a is a unit i.e. there is a $b \in A$ such that $ab = 1$. In particular, we get $\overline{a}\overline{b} = \overline{1}$ in $A/\mathfrak{N}$. So $A/\mathfrak{N}$ is a field.

(iii)$\Rightarrow$(i) Since $A/\mathfrak{N}$ is a field, $\mathfrak{N} \subset A$ is a maximal ideal. Let $\mathfrak{p} \subset A$ be prime ideal. By Proposition 7.2, $\mathfrak{N} \subset \mathfrak{p}$ thus ($\mathfrak{N}$ maximal) $\mathfrak{p} = \mathfrak{N}$. So $\mathfrak{N}$ is the only prime ideal of A.

Exercise 24. (Radical $\sqrt{\mathfrak{a}}$)

1. Since any $x \in \mathfrak{a}\mathfrak{b}$ can be written, $x = \sum_i a_i b_i$ with $a_i \in \mathfrak{a}$ and $b_i \in \mathfrak{b}$ we have $x \in \mathfrak{a}$ and $x \in \mathfrak{b}$ i.e. $\mathfrak{a}\mathfrak{b} \subset \mathfrak{a} \cap \mathfrak{b}$. Thus $\{\mathfrak{p} \in Spec(A),\ \mathfrak{a} \cap \mathfrak{b} \subset \mathfrak{p}\} \subset \{\mathfrak{p} \in Spec(A),\ \mathfrak{a}\mathfrak{b} \subset \mathfrak{p}\}$, hence (using Corollary 7.6) $\sqrt{\mathfrak{a}\mathfrak{b}} = \bigcap_{\mathfrak{a}\mathfrak{b} \subset \mathfrak{p}} \mathfrak{p} \subset \bigcap_{\mathfrak{a}\cap\mathfrak{b} \subset \mathfrak{p}} \mathfrak{p} = \sqrt{\mathfrak{a} \cap \mathfrak{b}}$.

   Take a $\mathfrak{p} \in Spec(A)$ such that $\mathfrak{a}\mathfrak{b} \subset \mathfrak{p}$. If $\mathfrak{a} \not\subset \mathfrak{p}$, pick $a \in \mathfrak{a}\backslash\mathfrak{a}\cap\mathfrak{p}$; then for any $b \in \mathfrak{b}$, since $ab \in \mathfrak{p}$ we have $b \in \mathfrak{p}$; so $\mathfrak{b} \subset \mathfrak{p}$. We get $\{\mathfrak{p} \in Spec(A),\ \mathfrak{a}\mathfrak{b} \subset \mathfrak{p}\} \subset \{\mathfrak{p} \in Spec(A),\ \mathfrak{a} \subset \mathfrak{p}\} \cup \{\mathfrak{p} \in Spec(A),\ \mathfrak{b} \subset \mathfrak{p}\}$; thus

   $$\sqrt{\mathfrak{a}} \cap \sqrt{\mathfrak{b}} = \bigcap_{\mathfrak{a} \subset \mathfrak{p}} \mathfrak{p} \cap \bigcap_{\mathfrak{b} \subset \mathfrak{p}} \mathfrak{p} = \bigcap_{\mathfrak{a} \subset \mathfrak{p} \text{ or } \mathfrak{b} \subset \mathfrak{p}} \mathfrak{p} \subset \bigcap_{\mathfrak{a}\mathfrak{b} \subset \mathfrak{p}} \mathfrak{p}\mathfrak{p} = \sqrt{\mathfrak{a}\mathfrak{b}}.$$

   Now for $x \in \sqrt{\mathfrak{a} \cap \mathfrak{b}}$, choose $k > 0$, such that $x^k \in \mathfrak{a} \cap \mathfrak{b}$; we have $x^{2k} = x^k \cdot x^k \in \mathfrak{a}\mathfrak{b}$. Therefore $x \in \sqrt{\mathfrak{a}\mathfrak{b}}$. Thus $\sqrt{\mathfrak{a} \cap \mathfrak{b}} = \sqrt{\mathfrak{a}\mathfrak{b}}$.

   Finally, for $x \in \sqrt{\mathfrak{a}\mathfrak{b}}$, choose $k > 0$, such that $x^k \in \mathfrak{a}\mathfrak{b}$. Since $\mathfrak{a}\mathfrak{b} \subset \mathfrak{a}$ and $\mathfrak{a}\mathfrak{b} \subset \mathfrak{b}$, we have $x^k \in \mathfrak{a}$ and $x^k \in \mathfrak{b}$ so $x \in \sqrt{\mathfrak{a}} \cap \sqrt{\mathfrak{b}}$. Thus $\sqrt{\mathfrak{a}} \cap \sqrt{\mathfrak{b}} = \sqrt{\mathfrak{a}\mathfrak{b}}$.

2. If $\sqrt{\mathfrak{a}} = (1)$ then there is a $n > 0$ such that $1 = 1^n \in \mathfrak{a}$ i.e. $\mathfrak{a} = (1)$. Of course, conversely if $\mathfrak{a} = (1) = A$, then $\mathfrak{a} \subset \sqrt{\mathfrak{a}} = A$.

3. Since by Corollary 7.6 $\sqrt{\mathfrak{a}}$ is an intersection of prime ideals $\sqrt{\mathfrak{a}} = \cap_{\mathfrak{a}\subset\mathfrak{p}}\mathfrak{p}$, if $\mathfrak{a} = \sqrt{\mathfrak{a}}$, $\mathfrak{a}$ is an intersection of prime ideals. Conversely, if $\mathfrak{a}$ is an intersection of prime ideals $\mathfrak{a} = \bigcap_{\mathfrak{p}\in I} \mathfrak{p}$, for some non empty subset $I \subset Spec(A)$. Now, take $x \in \sqrt{\mathfrak{a}}$ and $k > 0$ such that $x^k \in \mathfrak{a} = \bigcap_{\mathfrak{p}\in I} \mathfrak{p}$; then we have in particular $x^k \in \mathfrak{p}$ for any $\mathfrak{p} \in I$. Thus we get $x \in \mathfrak{p}$ for any $\mathfrak{p} \in I$, i.e. $x \in \bigcap_{\mathfrak{p}\in I} \mathfrak{p} = \mathfrak{a}$, proving $\sqrt{\mathfrak{a}} \subset \mathfrak{a}$ (hence $\mathfrak{a} = \sqrt{\mathfrak{a}}$).

Exercise 25. (Faithfully flatness)
Let $M \simeq \oplus_{i\in I} A$ be a free A-module and $f \in \mathrm{Hom}_A(N_1, N_2)$ such that $f \otimes_A \mathrm{id}_M = 0$. Fix a $i \in I$. For any $n_1 \in N_1$, then $0 = f \otimes_A \mathrm{id}_M(n_1 \otimes 1) = f(n_1) \otimes_A 1$ i.e. $f(n_1) = 0$ (we have a homomorphism of A-module $N_2 \otimes_A A \simeq N_2$, $n \otimes a \mapsto an$). Thus $f = 0$. So $\mathrm{Hom}_A(N_1, N_2) \to \mathrm{Hom}_A(M \otimes N_1, M \otimes N_2)$ is injective.
A module is projective if and only if it is a direct summand of a free module (see Solution to Exercise 12). And a direct summand of a flat (a fortiori a free module) module is flat (e.g. because tensor product commutes with direct sum). So a projective module is flat.
We have seen that for the product ring $A = k[x]/(f) \times k$, where $f \in k[x]$ is a polynomial of degree > 0, and the A-module $M = 0 \times k \subset A$, M is a projective module (as direct summand of the free module A). But $\mathrm{Hom}_A(A, A) \to \mathrm{Hom}_A(\underbrace{M \otimes A}_{\simeq M}, M \otimes A)$ is not injective as seen by the image of "the projection (not exactly) to the first summand $p : A \to A$, $(r, a) \mapsto (r, 0)$. We have $p \otimes \mathrm{id}_M((r, a) \otimes_A (0, t)) = (r, 0) \otimes (0, t) = (r, 0) \cdot (0, t) \otimes 1 = 0$.