Solutions for exercises, Algebra I (Commutative Algebra) – Week 13

Exercise 65. (Dimension)

1. $k[x,y]_{(x,y)}/(x^2-y^3)$: We have $dim(k[x,y]_{(x,y)}) = 2$ (and is an integral domain) so that according to Corollary 18.24, $dim(k[x,y]_{(x,y)}/(x^2-y^3)) = 1$. The maximal ideal of $k[x,y]_{(x,y)}/(x^2-y^3)$ is $(x,y)_{(x,y)}/(x^2-y^3)$ which is generated by $\overline{x}$ and $\overline{y}$. If $\overline{x} \in (\overline{y})$ then we can write $fx = yg + (x^2-y^3)h$ for a $f \notin (x,y)$ i.e. the constant term $f(0,0)$ of f is not 0. Evaluating the equality at $y=0$, we see that on the left hand side the coefficient $f(0,0)$ of x is non-zero and the right hand is in (x^2); contradiction. Likewise, we show that $\overline{y}$ is not in the ideal generated by $\overline{x}$. So $\overline{x}, \overline{y}$ is a minimal set of generators of $(x,y)_{(x,y)}/(x^2-y^3)$.

2. $k[x,y]_{(x,y)}/(x^2-y)$:We have $dim(k[x,y]_{(x,y)}) = 2$ so that according to Corollary 18.24, $dim(k[x,y]_{(x,y)}/(x^2-y)) = 1$. The maximal ideal of $k[x,y]_{(x,y)}/(x^2-y)$ is $(x,y)_{(x,y)}/(x^2-y)$ which is generated by $\overline{x}$; since $\overline{y} = \overline{x}^2 \in (\overline{x})$. It is necessary a minimal set of generators.

3. $k[x,y]_{(x,y)}/(x^2,y^3)$: We have $dim(k[x,y]_{(x,y)}) = 2$. Moreover $\sqrt{(x^2,y^3)} = (x,y)$ which is maximal. So (x^2,y^3) is a primary ideal and by Corollary 18.26, $dim(k[x,y]_{(x,y)}/(x^2,y^3)) = 2-2 = 0$.

4. $k[x,y,z]_{(x,y,z)}/(x^2+y^2+z)$: We have $dim(k[x,y,z]_{(x,y,z)}) = 3$ so that according to Corollary 18.24, $dim(k[x,y,z]_{(x,y,z)}/(x^2+y^2+z^n)) = 2$ and the maximal ideal is generated by $\overline{x}, \overline{y}, \overline{z}$.
   $\overline{x} \notin (\overline{y}, \overline{z})$: otherwise, one would be able to write $fx = yg_1 + zg_2 + (x^2+y^2+z^n)g_3$ with $f(0,0,0) \neq 0$ (i.e. $f \notin (x,y,z)$). Evaluating at $y = 0 = z$ (we get polynomials in x), on the left hand side, the coefficient of x is non-zero and on the right hand side the polynomial is in (x^2). Likewise $\overline{y} \notin (\overline{x}, \overline{z})$.
   If $n = 1$, then $\overline{z} = -(\overline{x}^2 + \overline{y}^2)$ so that $\overline{x}, \overline{y}$ generate $(x,y,z)_{(x,y,z)}/(x^2+y^2+z^n)$.
   If $n \geq 2$, then $\overline{z} \notin (\overline{x}, \overline{y})$: otherwise, one could write $fy = xg_1 + yg_2 + (x^2+y^2+z^n)g_3$ with $f(0,0,0) \neq 0$. Evaluating at $x = 0 = y$, on the left hand side, the coefficient of z is non-zero whereas on the right hand side the polynomial is in the ideal (z^n). So $(\overline{x}, \overline{y}, \overline{z})$ is a minimal set of generators.

Exercise 66. (Height and dimension)
Set $A = k[x,y,z]_{(x,y,z)}/(xy,xz)$. Looking at Spec$(A)$ as $V((xy,xz)) \subset \text{Spec}(k[x,y,z]_{(x,y,z)})$ and since $(xy,xz) \subset (y,z)$ and $(y,z) \in \text{Spec}(k[x,y,z]_{(x,y,z)})$, $(\overline{y}, \overline{z})$ is a prime ideal. Likewise $(xy,xz) \subset (x)$ and $(x) \in \text{Spec}(k[x,y,z]_{(x,y,z)})$ so that $(\overline{x})$ is a prime ideal. Again since $(x,y) \in \text{Spec}(k[x,y,z]_{(x,y,z)})$ and $(xy,xz) \subset (x,y)$, $(\overline{x}, \overline{y})$ is in Spec(A). Finally, $(x,y,z) \in \text{Spec}(k[x,y,z]_{(x,y,z)})$ and $(xy,xz) \subset (x,y,z)$, $(\overline{x}, \overline{y}, \overline{z})$ is in Spec(A). Thus we have in A, the chain of prime ideals $(\overline{x}) \subset (\overline{x}, \overline{y}) \subset (\overline{x}, \overline{y}, \overline{z})$. The inclusions are strict, which proves that $dim(A) \geq 2$: assume $fy = xg + xy\alpha + xz\beta$, with $f(0,0,0) \neq 0$; evaluating at $x = 0$, we get a contradiction so $(\overline{x}) \subsetneq (\overline{x}, \overline{y})$. Likewise (evaluating an equality $fz = xg_1 + yg_2 + xy\alpha + xz\beta$, with $f(0,0,0) \neq 0$, at $x = 0 = y$), $\overline{z} \notin (\overline{x}, \overline{y})$.
Moreover, since $xy \in k[x,y,z]_{(x,y,z)}$ is not a zero divisor, we have $dim(k[x,y,z]_{(x,y,z)}/(xy)) = 2$ and since $k[x,y,z]_{(x,y,z)} \twoheadrightarrow A$ (i.e. Spec$(A) \subset \text{Spec}(k[x,y,z]_{(x,y,z)})$), $dim(A) \leq 2$. So

You can still hand in solutions, but they will not be (necessarily) corrected anymore.

$dim(A) = 2$.
On the other hand we have $V((x, y, xz)) = V((x) \cdot (y, z)) = V(x) \cup V(y, z)$; so (y, z) and $(x) \in$ Spec(are minimal primes containing (xy, xz) i.e. $(\overline{x})$ and $(\overline{y}, \overline{z})$ are minimal (associated, isolated) primes of A. In particular $\text{ht}((\overline{x})) = 0 = \text{ht}((\overline{y}, \overline{z}))$.
We have $A/(\overline{y}, \overline{z}) \simeq k[x, y, z]_{(x,y,z)}/(y, z) + (xy, xz) \simeq k[x, y, z]_{(x,y,z)}/(y, z)$; which is readily seen to have dimension 1 ((x, y, z) is a regular sequence and use Corollary 18.26). So we get $dim(A) = 2 > 0 + 1 = \text{ht}((\overline{y}, \overline{z})) + dim(A/(\overline{y}, \overline{z})$.

Exercise 67. (Fibre dimension)
The homomorphism $A \to B$ is indeed an inclusion: if $f \in k[x, y]$ is in $(yz-x)$ i.e. $f = (yz-x)g$, evaluating at $z = 0 = x$, we get $f(0, y) = 0$ i.e. $f \in (x) \subset k[x, y]$ so we can write $xf_1 = (yz - x)g$. But then we must have $x|g$ so we can write $f_1 = (yz - x)g_1$ for some g_1. So we can repeat the argument; hence by induction $f = ax^n = (yz - x)g_n$ $(a \in k)$ which is possible only if $a = 0$ and $g_n = 0$.

1. The contraction of $\mathfrak{q}$ in $k[x, y, z]$ is $(y, z) + (yz - x)$. For $f \in k[x, y] \cap (y, z) + (yz - x)$ then $f(x, y, 0) = f$ and f can be written $f = yg_1 + zg_2 + (yz - x)g_3$; evaluating at $z = 0$, we get $f(x, y, 0) = f = yg_1(x, y, 0) - xg_3(x, y, 0)$ i.e. $f \in (x, y)$.
2. We have the inclusions of prime ideals in A: $(0) \subset (x) \subset (x, y)$ so $\text{ht}((x, y)) = 2$. Likewise since $B \simeq k[y, z]$, $\text{ht}((\overline{y}, \overline{z})) = 2$.
3. We have $B_\mathfrak{q}/\mathfrak{p}B_\mathfrak{q} \simeq k[y, z]_{(y,z)}/(yz, y) \simeq k[y, z]_{(y,z)}/(y)$ so $dim(B_\mathfrak{q}/\mathfrak{p}B_\mathfrak{q}) = 1$.

Exercise 68. (Singular points and the Jacobi criterion)
We can define a linear map $\varphi : k[x_1, \dots, x_n]_\mathfrak{m} \to k^n$ by $\frac{f}{g} \mapsto (\frac{1}{g}\frac{\partial f}{\partial x_1}(a_1, \dots, a_n), \dots, \frac{1}{g}\frac{\partial f}{\partial x_1}(a_1, \dots, a_n))$ (since $f(a_1, \dots, a_n) = 0$ and $g(a_1, \dots, a_n) \neq 0$). For any i, we have $\varphi(fracx_i - a_i1) = (0, \dots 0, 1, 0 \dots 0)$ the i^{th} vector of the canonical basis of k^n. So $\varphi_{|\mathfrak{m}}$ is surjective. Moreover for any i, j,

$$\varphi(\frac{(x_i - a_i)}{1}\frac{(x_j - a_j)}{1}) = (0, \dots, 0, \underbrace{a_i - a_i}_{j^{th}\text{ component}}, 0 \dots, 0, \underbrace{a_j - a_j}_{i^{th}\text{ component}}, 0, \dots 0) = 0$$

so $\mathfrak{m}^2 \subset \ker(\varphi_{|\mathfrak{m}})$ and we get an induced surjective map $\overline{\varphi} : \mathfrak{m}_\mathfrak{m}/\mathfrak{m}_\mathfrak{m}^2 \to k^n$.
But since Spec($k[x_1, \dots, x_n]$) is regular of dimension n, $dim_k(\mathfrak{m}_\mathfrak{m}/\mathfrak{m}_\mathfrak{m}^2) = n$ (see Example 18.27 and Prop. 18.28); thus $\overline{\varphi}$ is an isomorphism. Notice that $k[x_1, \dots, x_n]_\mathfrak{m} \simeq (k[x_1, \dots, x_n]/\mathfrak{m})_\mathfrak{m} \simeq (k)_\mathfrak{m} \simeq (k)_{(0)} \simeq k$.
By definition, the point $\mathfrak{m} \in V(f)$ is singular if and only if $\overline{\varphi}(\frac{f}{1}) = 0$.
According to Corollary 18.24, $dim(k[x_1, \dots, x_n]_\mathfrak{m}/(f)) = n - 1$ and its maximal ideal is $\mathfrak{m}_\mathfrak{m}/(f)$. So $\mathfrak{m}_\mathfrak{m}/(f)/(\mathfrak{m}_\mathfrak{m}/(f))^2 \simeq \mathfrak{m}/((f) + \mathfrak{m}_\mathfrak{m}^2)$. To see the isomorphism, start with the (obviously) surjective $p : \mathfrak{m}_\mathfrak{m} \to \mathfrak{m}_\mathfrak{m}/(f)/(\mathfrak{m}_\mathfrak{m}/(f))^2$ and notice that its kernel is exactly $\mathfrak{m}_\mathfrak{m} + (f)$.
Thus if $\mathfrak{m} \in V(f)$ is singular, then $\overline{\varphi}(f) = 0$, which since $\overline{\varphi}$ is an isomorphism, means $\frac{f}{1} = 0 \in \mathfrak{m}_\mathfrak{m}/\mathfrak{m}_\mathfrak{m}^2$ i.e. $\frac{f}{1} \in \mathfrak{m}_\mathfrak{m}^2$. So $(\frac{f}{1}) + \mathfrak{m}_\mathfrak{m}^2 = \mathfrak{m}_\mathfrak{m}^2$ so that $dim_k(\mathfrak{m}_\mathfrak{m}/(f)/(\mathfrak{m}_\mathfrak{m}/(f))^2) = dim_k(\mathfrak{m}/\mathfrak{m}^2) = n > dim(k[x_1, \dots, x_n]_\mathfrak{m}/(f))$ i.e. $k[x_1, \dots, x_n]_\mathfrak{m}/(f)$ is not regular.
Conversely, according to Corollary 18.12 we have $dim_k(\mathfrak{m}_\mathfrak{m}/(f)/(\mathfrak{m}_\mathfrak{m}/(f))^2) \geq dim(k[x_1, \dots, x_n]_\mathfrak{m}/(f)) = n-1$, so if $k[x_1, \dots, x_n]_\mathfrak{m}/(f)$ not regular, $dim_k(\mathfrak{m}_\mathfrak{m}/(f)/(\mathfrak{m}_\mathfrak{m}/(f))^2) \geq n$. Using $\mathfrak{m}_\mathfrak{m}/(f)/(\mathfrak{m}_\mathfrak{m}/(f))^2 \simeq \mathfrak{m}_\mathfrak{m}/((f)+\mathfrak{m}_\mathfrak{m}^2)$, this happens only if $(\frac{f}{1}) \subset \mathfrak{m}_\mathfrak{m}^2$; in which case $\frac{f}{1} = 0 \bmod \mathfrak{m}_\mathfrak{m}^2$ so $\overline{\varphi}(\frac{f}{1}) = \overline{\varphi}(0) = 0 = (\frac{\partial f}{\partial x_1}(a_1, \dots, a_n), \dots, \frac{\partial f}{\partial x_1}(a_1, \dots, a_n))$.