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Exercises, Algebra I (Commutative Algebra) – Week 12

Exercise 61. (Graded rings and modules, 3 points)
(i) Let $A = \bigoplus A_n$ be a graded ring and $a_i \in A_+$ homogenous elements. Then the a_i, $i \in I$, generate A as an A_0-algebra, i.e. $A = A_0[a_i]_{i\in I}$, if and only if they generate A_+ as an ideal, i.e. $A_+ = (a_i)_{i\in I}$.

(ii) Assume A is a graded ring such that A is a finitely generated A_0-algebra. If $M = \bigoplus_{n\in\mathbb{Z}} M_n$ is a graded A-module, which is a finite A-module in the usual sense, then each M_n is a finite A_0-module, see Remark 16.7.

Exercise 62. (Homogeneous ideals, 2 points)
Let A be a graded ring and $\mathfrak{a} \subset A$ a homogeneous ideal.

1. Show that $A/\mathfrak{a}$ is a graded ring.
2. Show $\sqrt{\mathfrak{a}}$ is a homogeneous ideal.

Exercise 63. (Proj, 5 points)

(i) For a graded ring A, show that $\mathrm{Proj}(A) = \emptyset$ if and only if every element in A_+ is nilpotent.

(ii) Show that $\mathbb{P}^0_k = \mathrm{Proj}(k[x])$ consists of just one point (namely the point corresponding to the zero ideal).

(iii) Show that for an algebraically closed field k there is a natural bijection between the set of closed points in $\mathbb{P}^n_k$ and the set

$$(\{(a_0,\ldots,a_n) \mid a_i \in k\} \setminus \{(0,\ldots,0)\}) / \sim,$$

where $\sim$ is defined by $(a_0,\ldots,a_n) \sim (\lambda a_0,\ldots,\lambda a_n)$ for all $\lambda \in k^*$. The map is given by $(a_0,\ldots,a_n) \mapsto (a_i x_j - a_j x_i)_{i,j=0,\ldots,n}$.

(iv) Show that $\mathfrak{p} \mapsto \mathfrak{p} \cap A_0$ defines a continuous map

$$\mathbb{P}^n_{A_0} := \mathrm{Proj}(A_0[x_0,\ldots,x_n]) \to \mathrm{Spec}(A_0).$$

Exercise 64. (Numerical polynomials, 4 points)
A polynomial $P \in \mathbb{Q}[T]$ is called *numerical* if $P(n) \in \mathbb{Z}$ for all $n \gg 0$. Prove the following assertions:

(i) If $P \in \mathbb{Q}[T]$ is a numerical polynomial of degree r, then there exist $c_0,\ldots,c_r \in \mathbb{Z}$ such that

$$P(T) = c_0\binom{T}{r} + c_1\binom{T}{r-1} + \ldots + c_r,$$

where $\binom{T}{k} = \frac{T(T-1)\ldots(T-k+1)}{k!}$.

Solutions to be handed in before Monday July 6, 4pm.

(ii) Assume $f\colon \mathbb{Z} \to \mathbb{Z}$ is such that the induced *difference function*

$$\Delta f\colon \mathbb{Z} \to \mathbb{Z},\ n \mapsto f(n+1) - f(n)$$

is polynomial, i.e. there exists a (numerical) polynomial $Q \in \mathbb{Q}[T]$ with $\Delta f(n) = Q(n)$ for $n \gg 0$. Show that then also f is polynomial, i.e. there exists a (numerical) polynomial $P \in \mathbb{Q}[T]$ with $f(n) = P(n)$ for $n \gg 0$. Moreover, $\deg P(T) = \deg Q(T) + 1$.

Exercise 65. (Grothendieck group, 5 points)
The *Grothendieck group* of an abelian category $\mathcal{C}$ is the quotient of the free abelian group generated by the objects of $\mathcal{C}$ by the equivalence relation given by short exact sequences:

$$K(\mathcal{C}) = \left\{ \sum_{i=1}^{n} a_i[M_i] \mid M_i \in \mathrm{Ob}(\mathcal{C}), a_i \in \mathbb{Z} \right\} /_{\sim},$$

where $[M_2] \sim [M_1] + [M_3]$ whenever there exists a short exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$. The interesting example for us is $\mathcal{C} = \mathrm{mod}(A_0)$.

(i) Show that the datum of an additive (in short exact sequences) function λ on $\mathcal{C}$ (see Definition 16.10) is equivalent to a group homomorphism $\lambda\colon K(\mathcal{C}) \to \mathbb{Z}$.

(ii) Show that $K(\mathrm{Vec}_{\mathrm{fd}}(k)) \cong \mathbb{Z}$.

(iii) Imitate the proof of Proposition 16.13 (lecture on Thursday) and show that for a finite graded module $M = \bigoplus_{n\geq 0} M_n$ over a Noetherian graded ring A the Poincaré series $P(M,t) = \sum_{n=0}^{\infty}[M_n]t^n \in K(\mathcal{C})[[t]]$ is of the form $f(t)/\prod(1-t^{d_i})^{-1}$ with $f(t) \in K(\mathcal{C})[t]$.