Prof. Dr. Daniel Huybrechts
Dr. René Mboro Summer term 2020

Exercises, Algebra I (Commutative Algebra) – Week 1

The first two exercise sheets will only use material you should be familiar with already. Some of it is covered and recalled by the first three lectures. These two sheets are not compulsory but the points can be counted towards your final score of the necessary 50% to get admitted to the exams.

Exercise 1. (Examples of rings, 3 points)

(i) Give an example (and prove that it satisfies the requirements) of a ring that is finite and non-commutative.

(ii) Let X be any set and A any ring. Show that the set of all maps from X to A:

$$A^X := \mathrm{Maps}(X, A) := \{f : X \to A\}$$

can be naturally endowed with the structure of a ring.

(iii) Let G be an abelian group and let $\mathrm{End}(G)$ be the set of all group homomorphisms $f : G \to G$. Show that $\mathrm{End}(G)$ can be naturally endowed with the structure of a ring. Is it (ever) commutative?

Exercise 2. (Examples of ring homomorphisms, 3 points)

(i) For a field k, consider

$$\phi : k[x] \to \mathrm{Maps}(k, k),\ f(x) \mapsto (a \mapsto f(a)).$$

Show that this is a ring homomorphism. Given a $a \in k$, show that $I_a := \{f \in \mathrm{Maps}(k, k),\ f(a) = 0\}$ is an ideal of $\mathrm{Maps}(k, k)$. Does $\phi^{-1}(I_a)$ have a particular structure in general?

(ii) For a set X with a distinguished element $x \in X$ and a ring A consider

$$\mathrm{Maps}(X, A) \to A,\ f \mapsto f(x).$$

Show that this is a ring homomorphism.

(iii) Can you describe a ring homomorphism $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}$ for some $n \neq 0$?

Exercise 3. (Kernels of ring homomorphisms, 3 points)
Consider the ring homomorphism of the previous exercise

$$\phi : k[x] \to \mathrm{Maps}(k, k),\ f(x) \mapsto (a \mapsto f(a))$$

for a field k. Prove that ϕ is injective if and only if k is infinite and in the case of finite fields, prove that the kernel of ϕ is a principal ideal.

Exercise 4. (Rings with few ideals, 4 points)

1. Every ring has at least two different ideals. Characterize those commutative rings that have precisely two ideals. Show that a commutative ring that is not a field contains a principal ideal $\neq (0), (1)$.

2. For k a field of characteristic 0 consider the ring $\mathrm{End}_k(k[x])$ of k-linear endomorphisms of the polynomial ring $k[x]$ (endowed with the ring structure considered in item (iii) of Exercise 1). Show that the differentiation map D has a right inverse but no left inverse.

Please turn over.

Solutions to be handed in before Monday April 13, 4pm.

– Until further notice the teaching for this course will be online via eCampus.

– Exercises can be downloaded from:
`http://www.math.uni-bonn.de/people/mboro/docs/comm_alg_SS20.htmpl`

– Solutions have to be handed in the following Monday before 4pm to the tutor of your group.

– Exams: first date - July 24, 2020; second date - September 21, 2020