# Arctic phenomena in random tilings with fixed boundaries

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# Outline

1. Random tilings and the Arctic phenomenon
2. Variational principle
3. Dimension 2: rhombus and domino tilings
4. Dimension 3: rhombohedra tilings

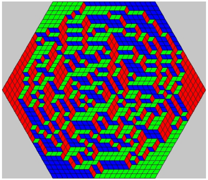
# Collaborators

- Mike Widom (Pittsburgh, USA)
- Rémy Mosseri (Paris, France)
- Francis Bailly (Paris, France)

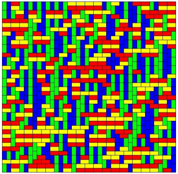
# 1 – Random tilings by rhombi, dominoes and rhombohedra

- finite set of prototiles (or tiles)
- covering of a compact region (*e.g.* of Euclidean space)
- no gaps or overlaps

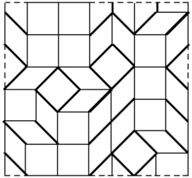
Examples:



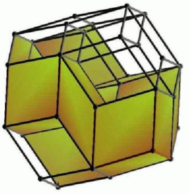
polygon rhombi



polygon dominoes



torus rhombi



polyhedron rhombohedra

# Tilings by rhombic tiles – Entropy and shape

$D \to d$ tilings:

- $d$-dimensional euclidean space
- $D$ edge orientations
- $\binom{D}{d}$ rhombic prototiles

Physical symmetries: octagonal ($4 \to 2$), decagonal ($5 \to 2$), icosahedral ($6 \to 3$)

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Questions:

1. How many tilings of a given region?
   Large size limit: $S = \lim_{N\to\infty} \frac{\log(\#\text{ tilings})}{N}$
2. What is the typical "shape" of a tiling?

The entropy per tile $S$ depends on tile fractions
*and boundary conditions*

# Example: hexagonal ($3 \to 2$) tilings

Equal tile fractions ("diagonal" case): $n_1 = n_2 = n_3 = \frac{1}{3}$

- Periodic (or free) boundary conditions (torus):
  $S = \frac{2}{\pi}\int_0^{\pi/3} \log(2\cos x)\, dx = 0.323\ldots$ (Wiannier 1950)
- Polygonal boundary conditions:
  $S = \frac{3}{2}\log 3 - 2\log 2 = 0.261\ldots$ (Elser 1985)

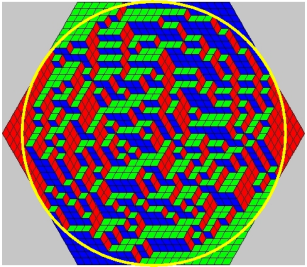
Analytically solved models (periodic boundaries; entropy maxima)

- squares-triangles (12-fold symmetry):
  $S = 0.120\ldots$ (per vertex; Widom; Kalugin, '93,'94)
- rectangles-triangles (8-fold symmetry):
  $S = 0.119\ldots$ (per area; de Gier, Nienhuis, '96)
- rectangles-triangles (10-fold symmetry):
  $S = 0.175\ldots$ (per vertex; de Gier, Nienhuis, '98)

# Arctic phenomenon

## Example: hexagonal (3 → 2) tilings

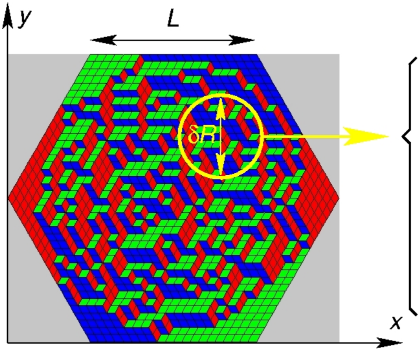
- frozen regions near the boundary
- gradient of entropy
- macroscopic effect on typical tilings (typical “shape”)
- macroscopic heterogeneity



1. What is the shape of the “arctic curve”?
2. What are the tile statistics inside the arctic curve?

# 2 – Variational principle

N. Destainville, R. Mosseri, F. Bailly, *J. Stat. Phys.* (1997)
N. Destainville, *J. Phys. A.* (1998)
H. Cohn, R. Kenyon, J. Propp, *J. Amer. Math. Soc.* (2001)



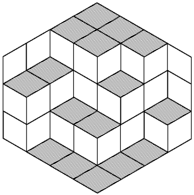

By contrast:

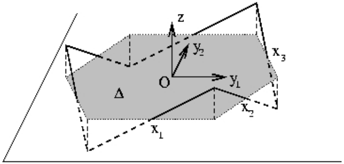
- Local patch of tiling $1 \ll \delta R \ll L \to \infty$
- locally homogeneous
- local tile fractions $n_1, n_2, n_3$
- local entropy per tile $\sigma(n_1, n_2, n_3)$: free-boundary entropy per area

Coarse-graining (or continuous limit): 3 regular functions

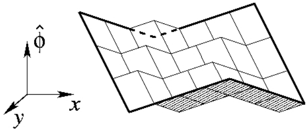
$$n = (n_1(x,y), n_2(x,y), n_3(x,y)) \text{ (such that } n_1 + n_2 + n_3 = 1)$$

# Height-function or directed-membrane representation

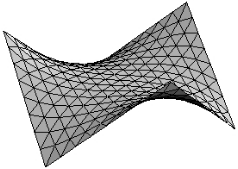





Boundary conditions




Height function: $\hat{\phi} : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$



Coarse-graining when $L \to \infty$

$\hat{\phi}$ facetted

$\phi$ smooth

- Coarse-graining $\equiv$ rescaling of factor $1/L$
- Tile side $= 1/L \to 0$ when $L \to \infty$
- Only large scale (macroscopic) fluctuations remain
- One-to-one correspondance $\nabla\phi \leftrightarrow$ tile fractions $n_1, n_2, n_3$
  $\sigma(n_1, n_2, n_3) = \sigma(\nabla\phi)$: free-boundary entropy per area
- **Entropy functional:** $\mathcal{N}_\phi$ = Number of $N$-tile facetted membranes $\hat{\phi}$ "close" to $\phi$ after rescaling
  $$s[\phi] = \lim_{N\to\infty} \frac{\log(\mathcal{N}_\phi)}{N}$$
  $s[\phi]$ accounts for the microscopic degrees of freedom
  $$s[\phi] = \frac{1}{V(\Delta)} \int_\Delta \sigma(\nabla\phi)\, dxdy$$
- Functional integral:
  $\mathcal{N}_{fixed}(N) \approx \sum_{\phi\in\Phi} \mathcal{N}_\phi = \int_\Phi \mathcal{D}\phi \; exp(Ns[\phi])$
  $S(N) = \log(\mathcal{N}_{fixed}(N))/N$

# Maximization of $s[\phi]$

Assume that:

- $s[\phi]$ has a unique maximum $\phi_{max} \in \Phi$
- $s[\phi]$ is regular (quadratic) near $\phi_{max}$

$\Rightarrow$ Saddle-point argument: $\lim_{N\to\infty} S(N) = s[\phi_{max}]$

The statistical ensemble is dominated by states "close" to $\phi_{max}$ at the large size limit

- relates (formally) the fixed-boundary entropy $S$ to the free-boundary one $\sigma$
- the knowledge of $\phi_{max}$ provides the tile statistics at each point $(x, y)$
- BUT: REQUIRES THE KNOWLEDGE OF $\sigma(\nabla\phi)\ldots$

# 3 – Dimension 2: hexagonal ($3 \to 2$) tilings

- $\sigma(\nabla\phi = (E_1, E_2))$ is known (Wannier, 1950)
- outside the arctic circle: $\phi_{max}$ affine: periodic tiling
- inside the circle:

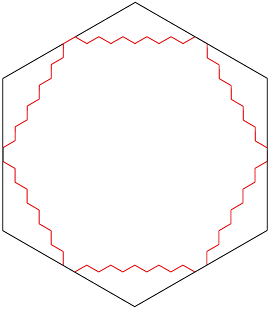
$$E_1 = \frac{3}{\pi\sqrt{2}}[\text{cotan}^{-1} f(x,y) + \text{cotan}^{-1} f(-x,y)] - \sqrt{2}$$

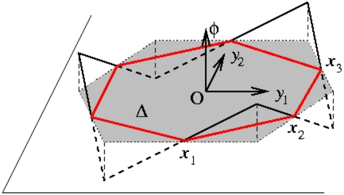
$$E_2 = \frac{\sqrt{3}}{\pi\sqrt{2}}[\text{cotan}^{-1} f(x,y) - \text{cotan}^{-1} f(-x,y)]$$

$$f(x,y) = \frac{1}{2\sqrt{3}} \frac{8/\sqrt{3}xy - 8/3y^2 + 2}{\sqrt{1 - 4/3(x^2 + y^2)}}$$

- Non-diagonal tilings $L_1 \neq L_2 \neq L_3$: circle $\to$ ellipse.

# Strain-free fixed boundaries: corrugated hexagon

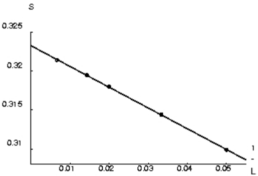




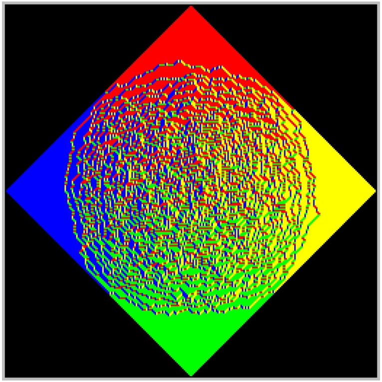

$S[\phi]$ is maximized by the constant function $\phi_{max} = 0$.
Homogeneous tiling, no frozen corners: $S = \sigma(\nabla\phi = 0) = S_{free}$

Check: exact enumeration at finite $L \leq 150$ by a determinental method (Gessel, Viennot, '85)

$S_{fit} = 0.32309$ and $S_{free} = 0.32307$

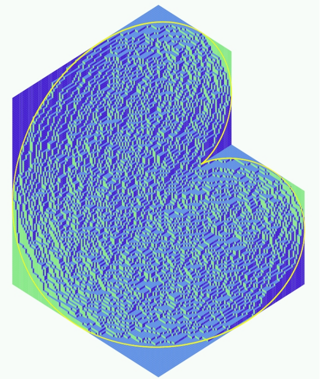


# Other examples



Dominoes in the Aztec Diamond
[Henry Cohn, Noam Elkies,
Jim Propp, 1996]

circle

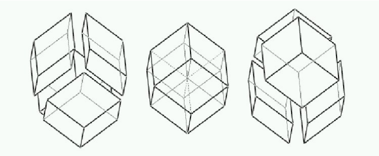


Rhombi in a truncated hexagon
[Richard Kenyon and Andrei Okounkov,
arXiv:math-ph/0507007]

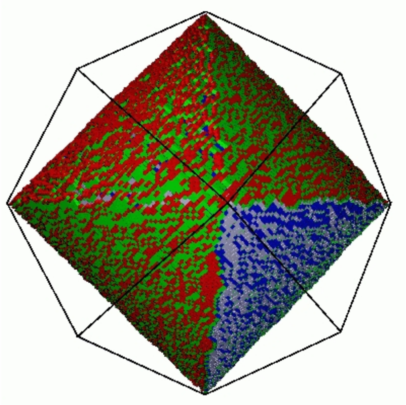
cardioid

$\Rightarrow$ connection with algebraic geometry

- 4 rhombohedral prototiles:

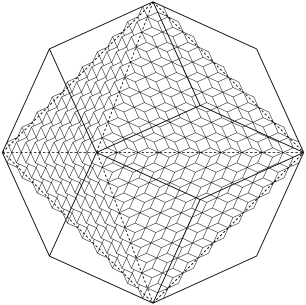


- boundary: rhombic dedecahedron of side $L$
- Direct observation [Linde, Moore, Nordhal, 2001]
- Arctic surface: octahedron



8 pyramidal frozen regions removed

Confirmation: Entropy calculations
[Widom, Mosseri, ND, Bailly, 2002]



$$\left.\begin{array}{l} S_{\text{``free''}} \simeq 0.214 \\ S_{fixed} \simeq 0.145 \end{array}\right\} \text{ratio } \simeq 1.48 \pm 0.03$$

Variational principle: 3/2 **if and only if**
(assuming uniqueness of $\phi_{max}$):

1. tiling frozen outside the octahedron
2. tiling homogeneous inside the octahedron

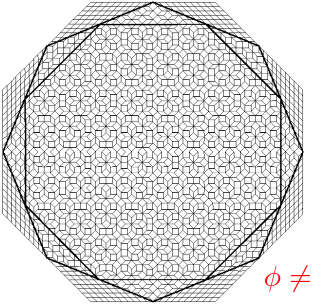
corrugated octahedron: $S = S_{free}$

**Consequence**: in 3D, the relationship between fixed- and free-boundary tilings is heighly simplified

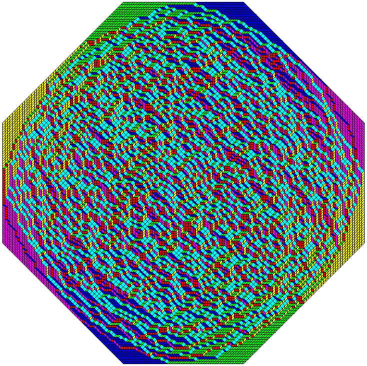
Fixed-boundary properties can be much more easily transposed to free-boundary tilings of physical interest

**Conjecture**: *in dimension 3 and above, arctic frontiers are polyhedra.*

# Open problem: octagonal symmetry and beyond



$\phi \neq \phi_{max}$



[figure from Matthew Blum]

- frozen outer crown
- $3 \to 2$ crown: effective $D = D' = 3$
- $4 \to 2$ central region

PROBLEM: $\sigma(\nabla\phi)$ unknown...